Navdeep Singh Dhindsa 15/02/2024

JC Talk

Lattice Gauge Theories And Tensors

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Tensor network representation of non-abelian gauge theory coupled to reduced staggered fermions

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(Dated: December 27, 2023)

Abstract

We show how to construct a tensor network representation of the path integral for reduced staggered fermions coupled to a non-abelian gauge field in two dimensions. The resulting formulation is both memory and computation efficient because reduced staggered fermions can be represented in terms of a minimal number of tensor indices while the gauge sector can be approximated using Gaussian quadrature with a truncation. Numerical results obtained using the Grassmann TRG algorithm are shown for the case of $SU(2)$ lattice gauge theory and compared to Monte Carlo results.

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Paper to discuss

Before we discuss Tensor networks ...

- Discretizing gauge theories on a spacetime lattice allows for powerful numerical simulations using MCMC methods.
- Despite the great success, infamous sign problem restricts its usage in certain coupling regimes.

Last JC talk by Sayak

Sign Problem

Say S_1, S_2 be two action values corresponding to two configurations C_1, C_2

If $S_1, S_2 \in \mathbb{R}$, we can comment on which is bigger e^{-S_1}, e^{-S_2}

If $S_1, S_2 \in \mathbb{C}$, can we comment on which is bigger ? e^{-S_1}, e^{-S_2}

Which contribute to path integral more ?

Sign Problem

$$S = \int d\tau \left(-\frac{1}{2}\phi\partial_\tau^2\phi + \overline{\psi}\partial_\tau\psi + \overline{\psi}W''(\phi)\psi + \frac{1}{2}\left[W'(\phi)\right]^2 \right)$$

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi\, e^{-S_B - S_F}$$

Integrating out fermions

$$\mathcal{Z} = \int \mathcal{D}\phi\, \det(M)\, e^{-S_B}$$

Effective Action

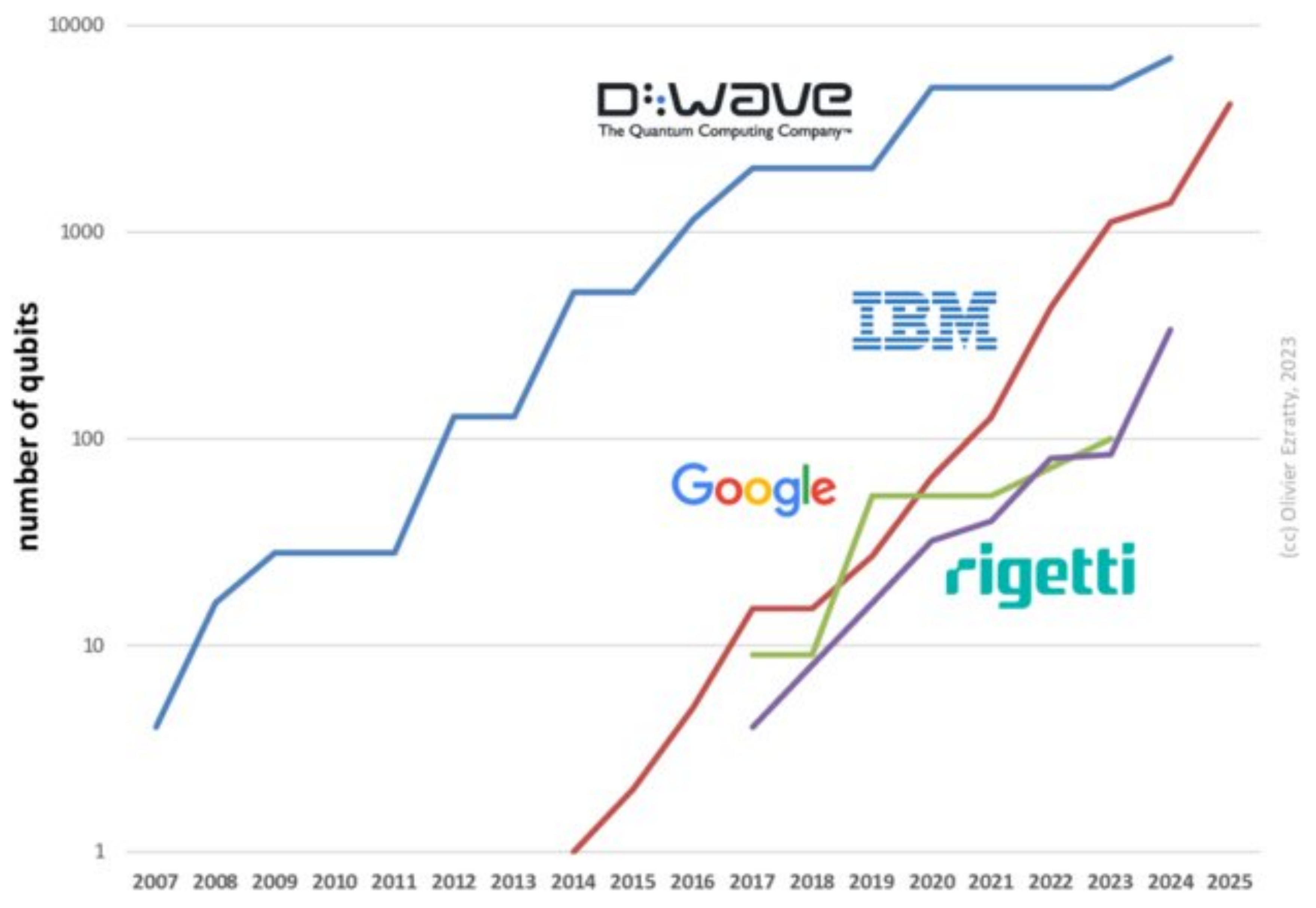
$$S = S_B - log(\det(M))$$

QCD at finite chemical potential

Credit: Olivier Ezratty

Alternatives

Quantum



Why Quantum Computation - Real time simulations with MC or classical simulation methods challenging

Qubit based approach highly used at the moment to study Lattice gauge theories using QC

How many qubits do we need to simulate even a sub parameter space of QCD ?

How long will it take to reach a sufficient number ?

Alternatives

Classical

- Phase quenched Monte Carlo
- Complex Langevin
- Lefschetz Thimble
- **Tensor Networks**

Interested in

Numerical Bootstrap ?

arXiv:2010.06539 [hep-lat]

Meurice, Sakai, Unmuth-Yockey

Tensor Networks

- ◉ Tool to represent wave functions of quantum many body systems.
- ◉ Partition functions of target models can be visualized as an assembly obtained:

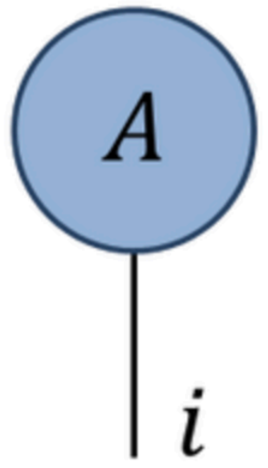
- by "wiring" together objects carrying multiple "legs"
- attached to the sites, links or plaquettes of a Euclidean space-time lattice

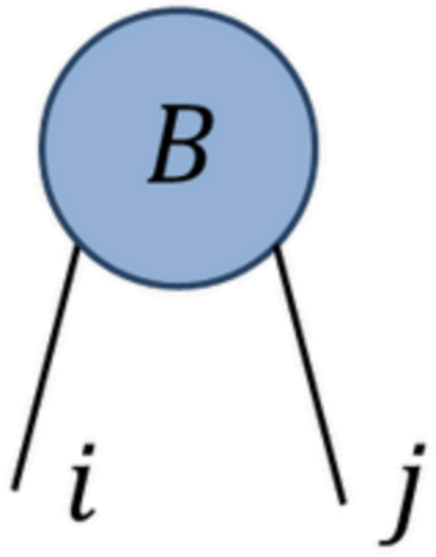
https://www.tensors.net/ — Introduction

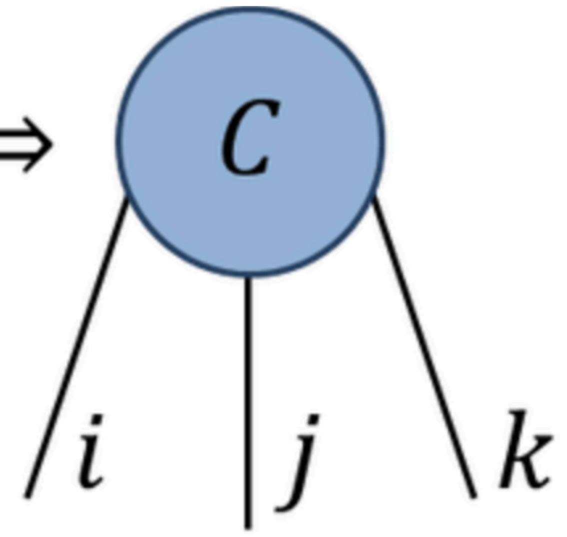
https://tensornetwork.org/trg/ — TRG (focus of the day)

Tensor Networks

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix} \qquad C = \begin{bmatrix} C_{111} & \cdots & C_{1n1} \\ \vdots & \ddots & \vdots \\ C_{m11} & \cdots & C_{mn1} \end{bmatrix}$$

$A_i \Leftrightarrow$ 

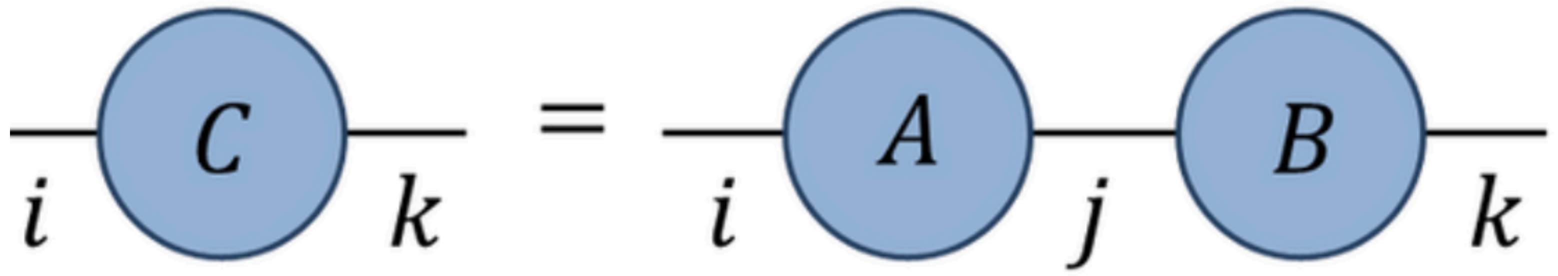
$B_{ij} \Leftrightarrow$ 

$C_{ijk} \Leftrightarrow$ 

Tensor representation

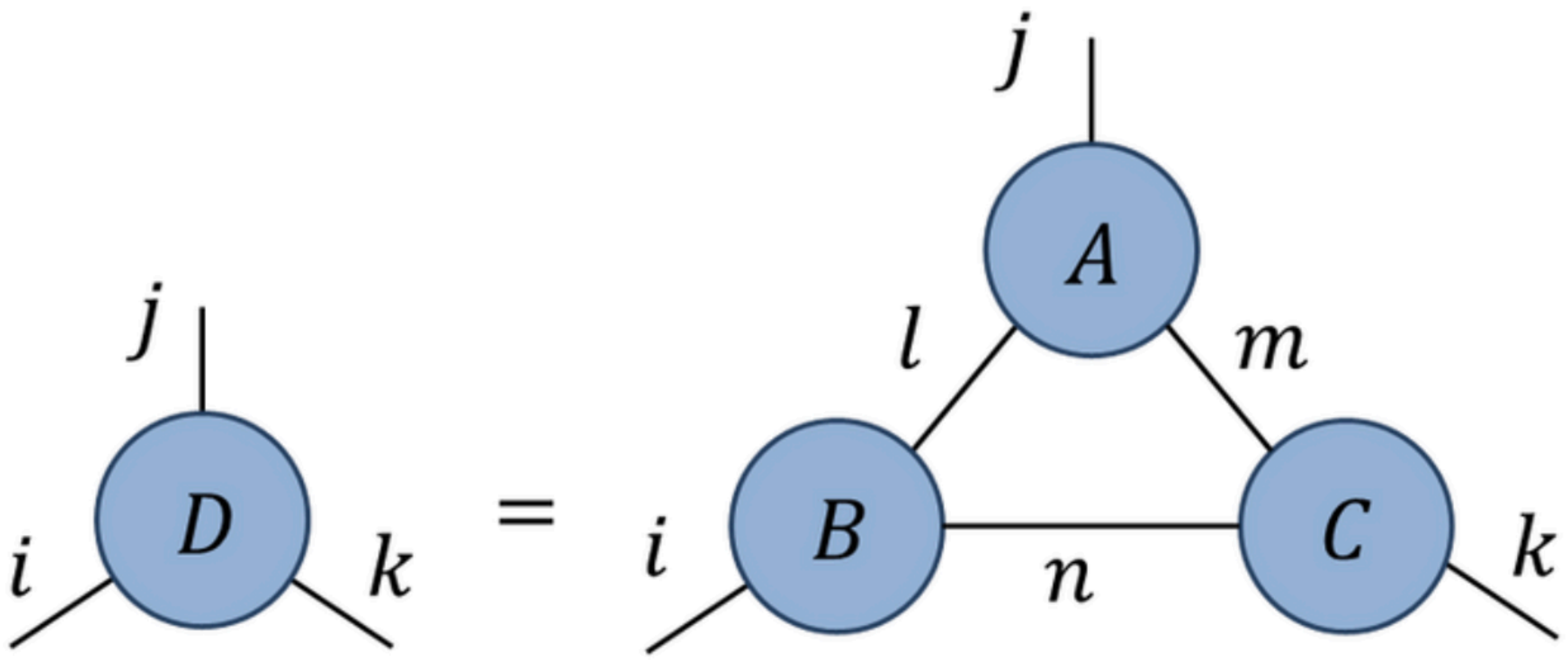
Tensor Networks

Contraction of Tensors



$$\Updownarrow$$

$$C_{ik} = \sum_{j} A_{ij} B_{jk}$$



$$\Updownarrow$$

$$D_{ijk} = \sum_{lmn} A_{ljm} B_{iln} C_{nmk}$$

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Tensor Networks

For 1+1 dimensional models various tensor network approaches have been used

MPS **DMRG** **TRG**

Scalability ....?

TRG

Renormalization Group: Systematic investigation of changes of physical system as viewed at different scales.

In simpler terms: Replacing the elementary degrees of freedom by new averaged variables at larger scales.

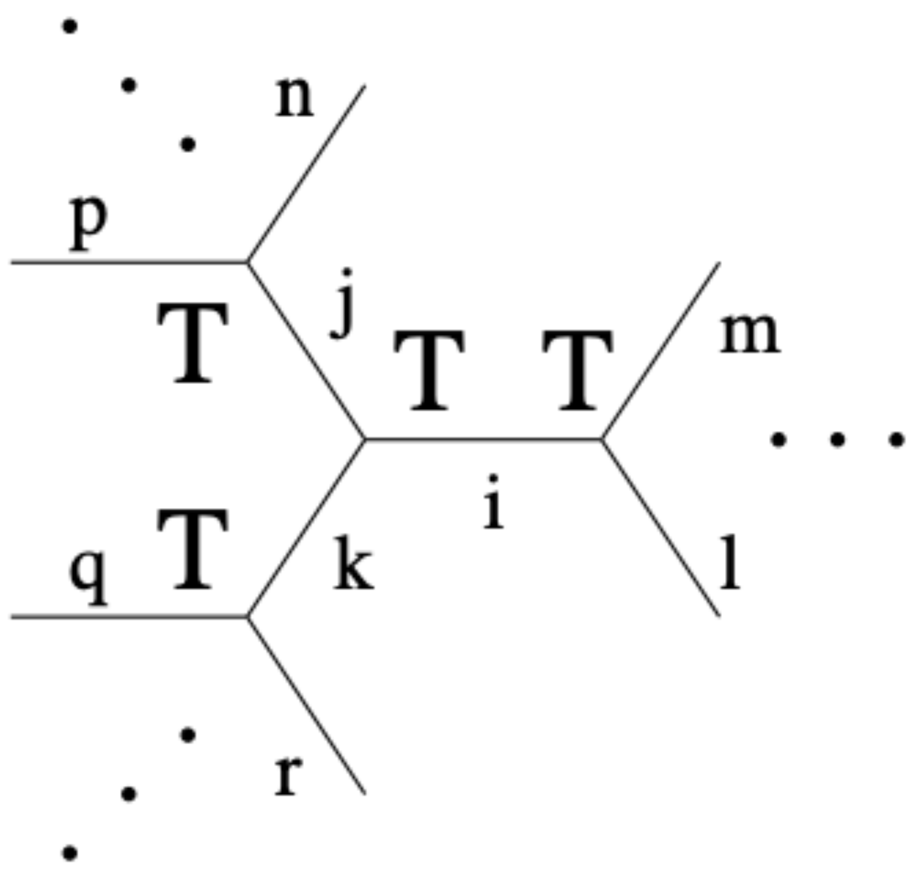
Performing the same procedure to compute partition function by recursive blocking of tensors —— TRG

TRG

arXiv:cond-mat/0611687 [cond-mat.stat-mech]

Levin, Nave

How to write partition function using tensor network algorithms ?



Honeycomb Lattice with

$i, j, k - 1,2,\ldots\ldots D$

Then weight of the configuration is given by:

$$e^{-S(i,j,k,\ldots)} = T_{ijk} T_{ilm} T_{jnp} T_{kqr} \cdots$$

TRG

arXiv:cond-mat/0611687 [cond-mat.stat-mech]

Levin, Nave

How to write partition function using tensor network algorithms ?

Partition function is obtained by taking the product of all the tensors, contracting the pairs of indices on each bond.

$$Z = \sum_{ijk\ldots} e^{-S(i,j,k,\ldots)} = \sum_{ijk\ldots} T_{ijk} T_{ilm} T_{jnp} T_{kqr} \cdots$$

Partition Function

Then weight of the configuration is given by:

$$e^{-S(i,j,k,\ldots)} = T_{ijk} T_{ilm} T_{jnp} T_{kqr} \cdots$$

TRG

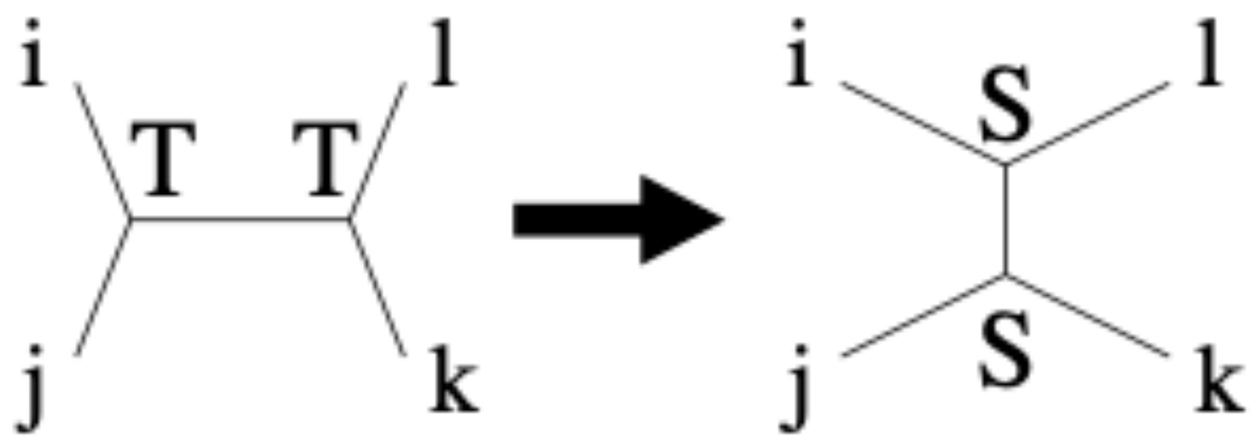
arXiv:cond-mat/0611687 [cond-mat.stat-mech]

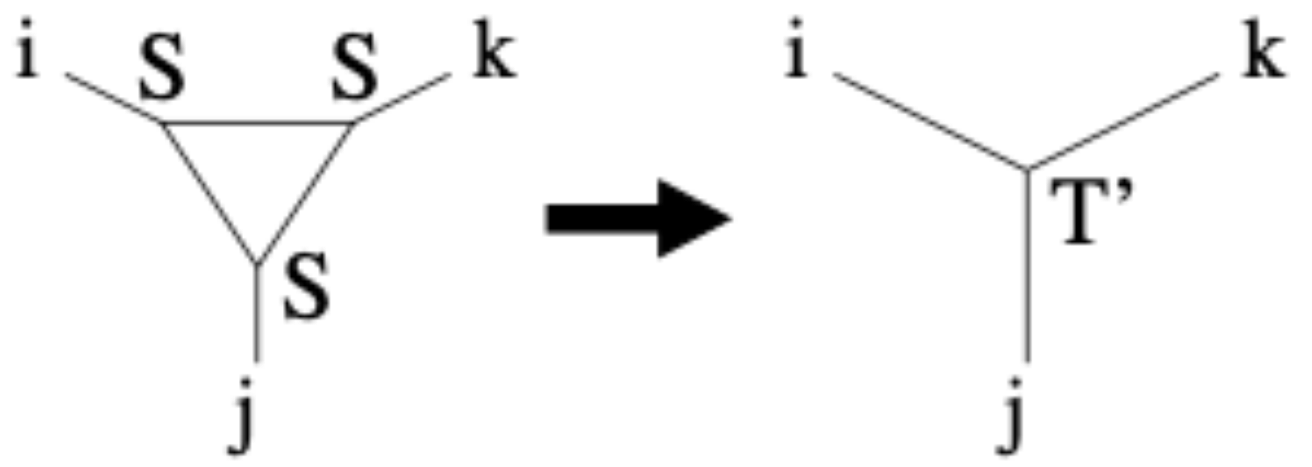
Levin, Nave

Each iteration consists of two steps

- Approximate (involving SVD)
- Exact (Contraction)

$$\sum_{n} S_{lin} S_{jkn} \approx \sum_{m} T_{ijm} T_{klm}$$





$$T'_{ijk} = \sum_{pqr} S_{kpq} S_{jqr} S_{irp}$$

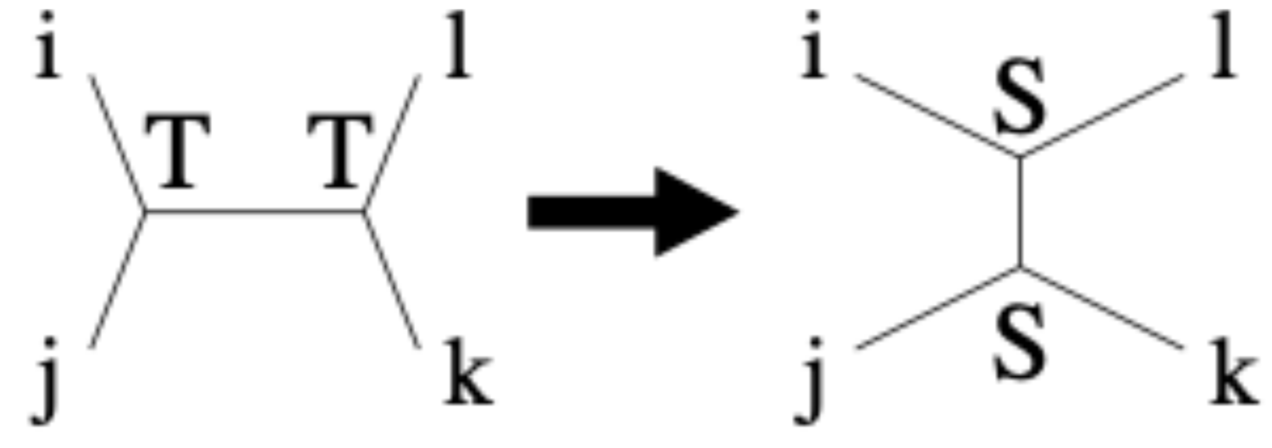
TRG

arXiv:cond-mat/0611687 [cond-mat.stat-mech]

Levin, Nave

- Approximate (involving SVD)

$$\sum_n S_{lin} S_{jkn} \approx \sum_m T_{ijm} T_{klm}$$



Why ?

$$D^2 \times D^2 \quad (TT) \to D^2 \times D \ (S)$$

Find a matrix M s.t. $M = S \,.\, S^T$

which is not straightforward as we are comparing matrices with rank D^2 and D

TRG

arXiv:cond-mat/0611687 [cond-mat.stat-mech]

Levin, Nave

- Approximate (involving SVD)

Find a matrix M s.t. $M = S \,.\, S^T$

which is not straightforward as we are comparing matrices with rank D^2 and D

$$M_{li,jk} = \sum_n s_n U_{li,n} V^*_{jk,n}$$

Truncate U and V upto maximum D singular values

$$S^A_{lin} = \sqrt{s_n}\tilde{U}_{li,n},\; S^B_{jkn} = \sqrt{s_n}\tilde{V}^*_{jk,n}$$

Credit: Vamika

TRG

- Approximate (involving SVD)

Marienplatz

Plaza in Munich, Germany

3825×4861 real matrix

$D_{\text{cut}} = 30$

$D_{\text{cut}} = 100$

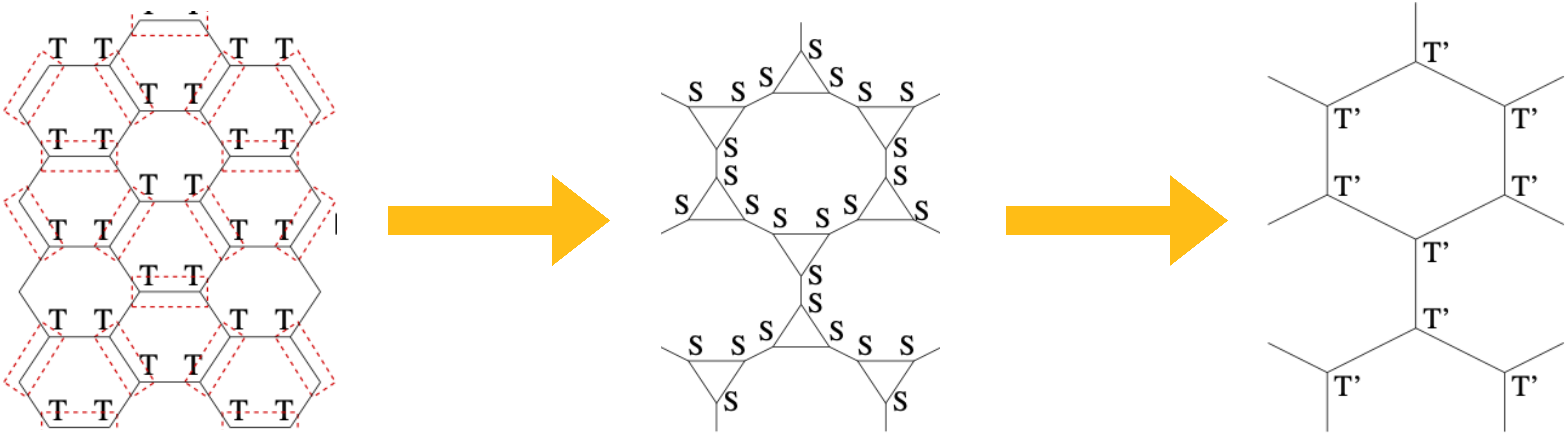
$D_{\text{cut}} = 300$

$D_{\text{cut}} = 1000$

TRG

arXiv:cond-mat/0611687 [cond-mat.stat-mech]

Levin, Nave



$$\sum_n S_{lin} S_{jkn} \approx \sum_m T_{ijm} T_{klm}$$

$$T'_{ijk} = \sum_{pqr} S_{kpq} S_{jqr} S_{irp}$$

TRG

- Iteratively proceed in same manner
- In this honeycomb example lattice points decreased by a factor of 3
- At last we are down to a single tensor
- Partition function is trace of this tensor

Scalability of this method - difficult though there exists other versions such as **HOTRG**

Non-Abelian Higgs Model in 2d

$$S_g = -\frac{\beta}{2} \sum_x \mathrm{Tr}\left[U_{x,1} U_{x+\hat{1},2} U^\dagger_{x+\hat{2},1} U^\dagger_{x,2}\right]$$

Wilson Action

$$S_\Phi = -\frac{\kappa}{2} \sum_x \sum_{\mu=1}^{2} \rho_{x+\hat{\mu}} \rho_x \,\mathrm{Tr}\left[\alpha^\dagger_{x+\hat{\mu}} U_{x,\mu} \alpha_x\right]$$

Gauge-Matter term

$$V = \sum_x \rho_x^2 + \lambda(\rho_x^2 - 1)^2$$

Potential term

Partition Function

$$Z = \int D[U] D[\rho] D[\alpha] e^{-S_g - S_\Phi - V}$$

Non-Abelian Higgs Model in 2d

Partition function in terms of character functions

$$f(X\,\mathrm{Tr}[V]) = \sum_{r=0}^{\infty} F_r(X)\chi^r(V)$$

$$\begin{aligned} e^{-S_g} &= \exp\left[\frac{\beta}{2}\sum_x \mathrm{Tr}[U_{x,1}U_{x+\hat{1},2}U^\dagger_{x+\hat{2},1}U^\dagger_{x,2}]\right] \\ &= \prod\sum F_r(\beta)\chi^r(U_{x,1}U_{x+\hat{1},2}U^\dagger_{x+\hat{2},1}U^\dagger_{x,2}) \end{aligned}$$

$$\begin{aligned} \chi^r(U_1U_2U_3\ldots U_n) &= D^r_{nn}(U_1U_2U_3\ldots U_n) \\ &= D^r_{ab}(U_1)D^r_{bc}(U_2)\ldots D^r_{za}(U_n) \end{aligned}$$

$$\begin{aligned} &\chi^r(U_{x,1}U_{x+\hat{1},2}U^\dagger_{x+\hat{2},1}U^\dagger_{x,2}) \\ &= D^r_{ab}(U_{x,1})D^r_{bc}(U_{x+\hat{1},2})D^{r\dagger}_{cd}(U_{x+\hat{2},1})D^{r\dagger}_{da}(U_{x,2}) \\ &= D^r_{ab}(U_{x,1})D^r_{bc}(U_{x+\hat{1},2})D^{r*}_{dc}(U_{x+\hat{2},1})D^{r*}_{ad}(U_{x,2}) \end{aligned}$$

Non-Abelian Higgs Model in 2d

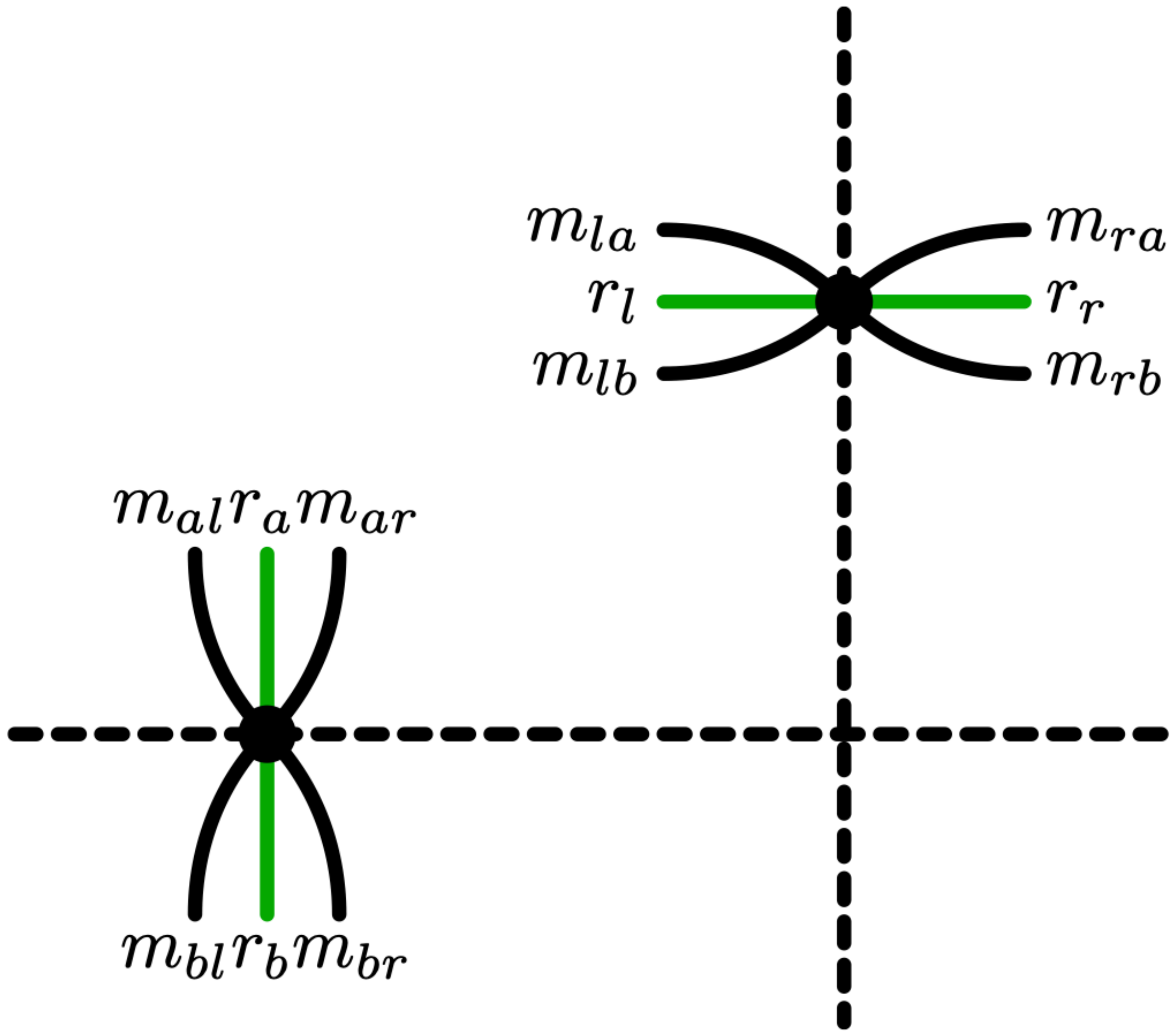
- In 2d, two plaquettes associated with single link
- Another additional variable from gauge-matter term
- Three matrices associated with each link

$$\sum_{n=-\sigma}^{\sigma} \int dU\, D^{r_1}_{m_1 n_1}(U) D^{r_2 \dagger}_{m_2 n_2}(U) D^{\sigma}_{nn}(U)$$

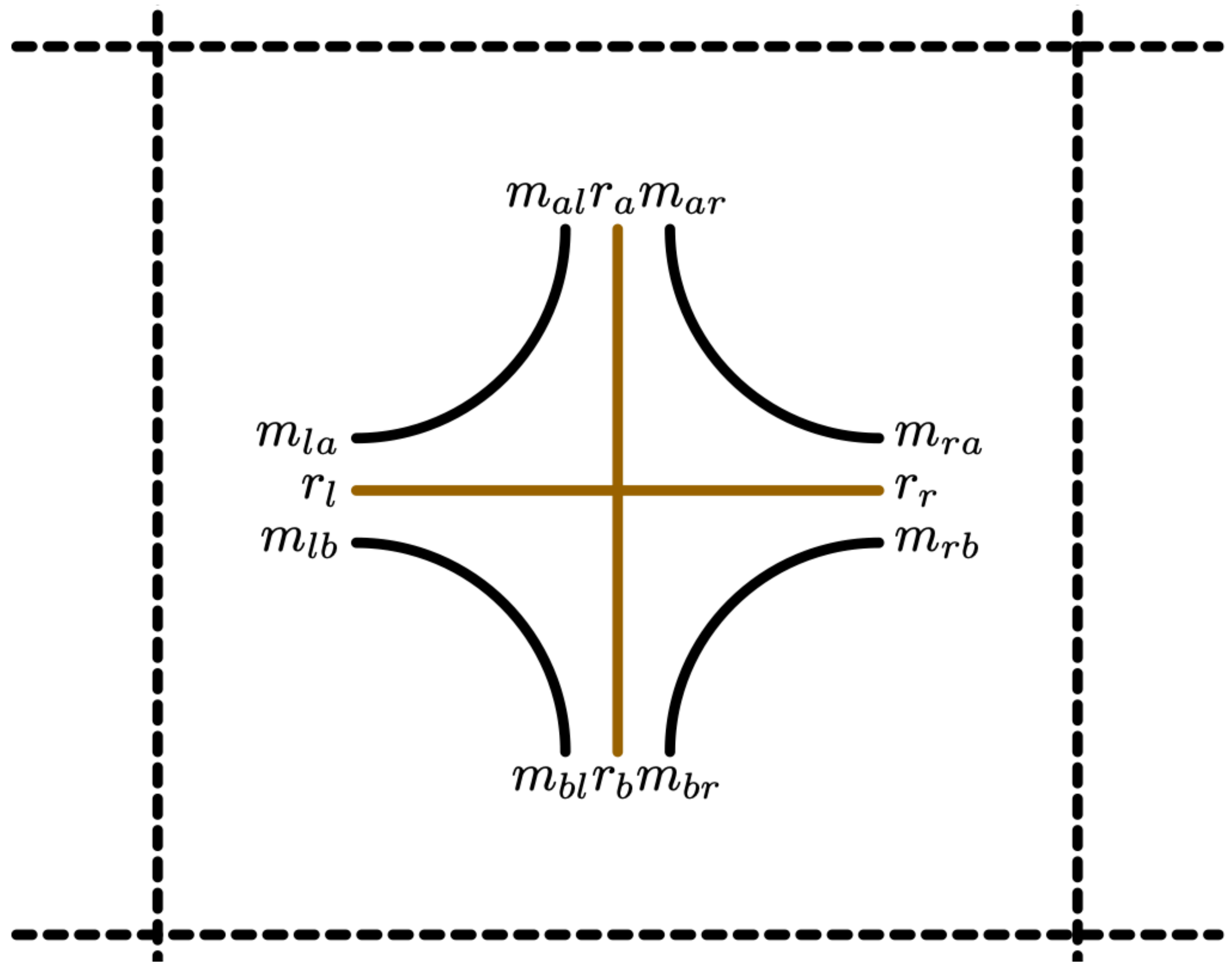
$$= \sum_{n=-\sigma}^{\sigma} \frac{1}{d_{r_2}} C^{r_2 n_2}_{r_1 m_1 \sigma n} C^{r_2 m_2}_{r_1 n_1 \sigma n}.$$

Integral over each link has form

Non-Abelian Higgs Model in 2d

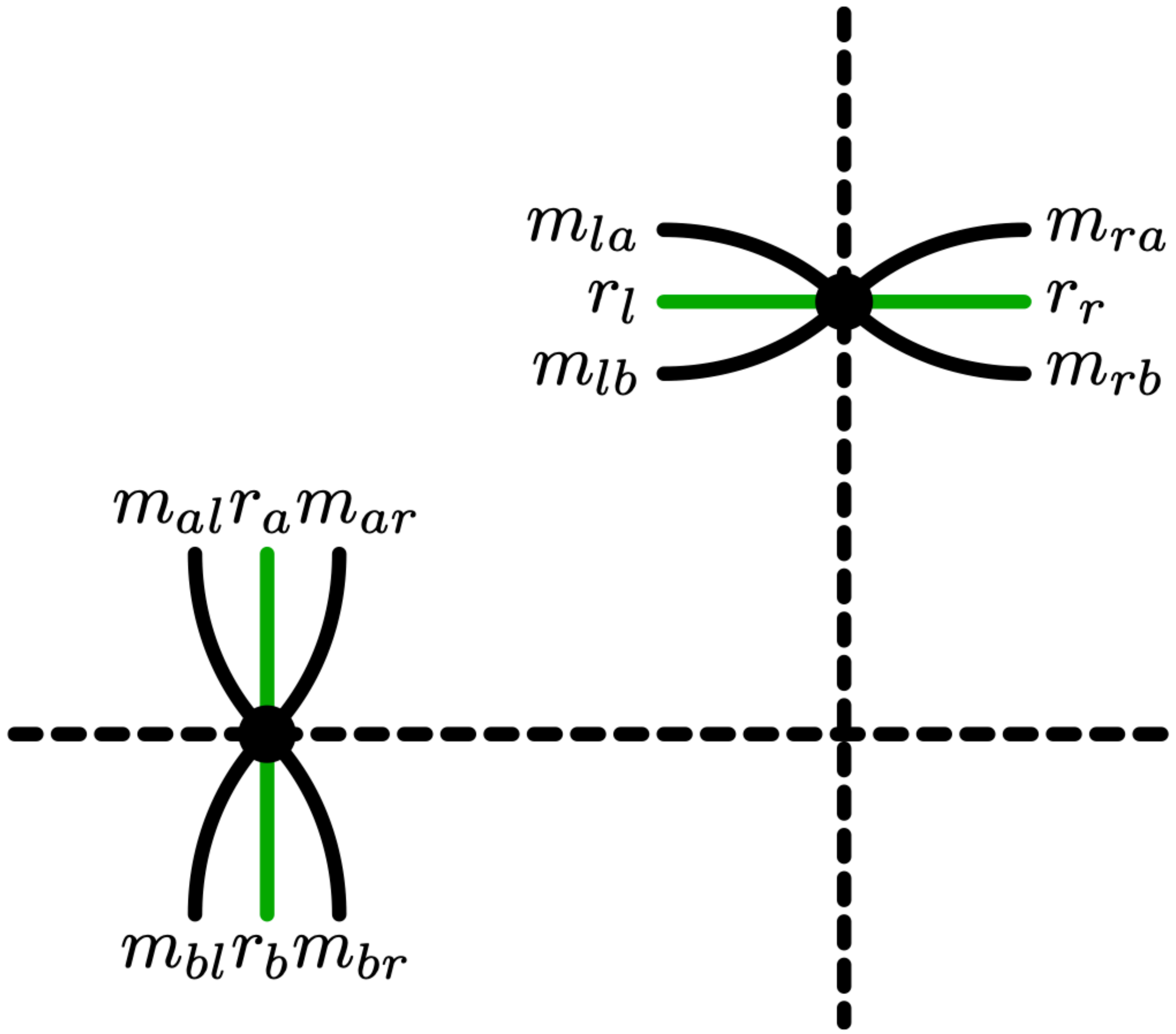


Tensors on link

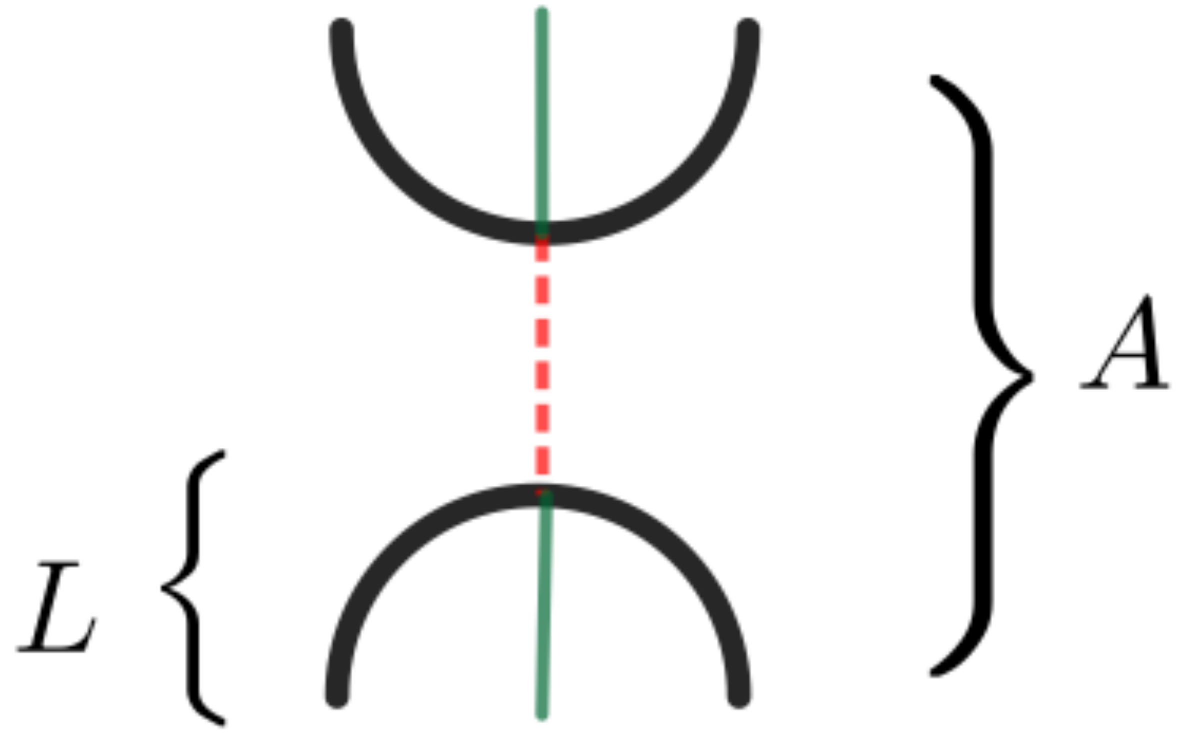


Tensor on Plaquette

Non-Abelian Higgs Model in 2d

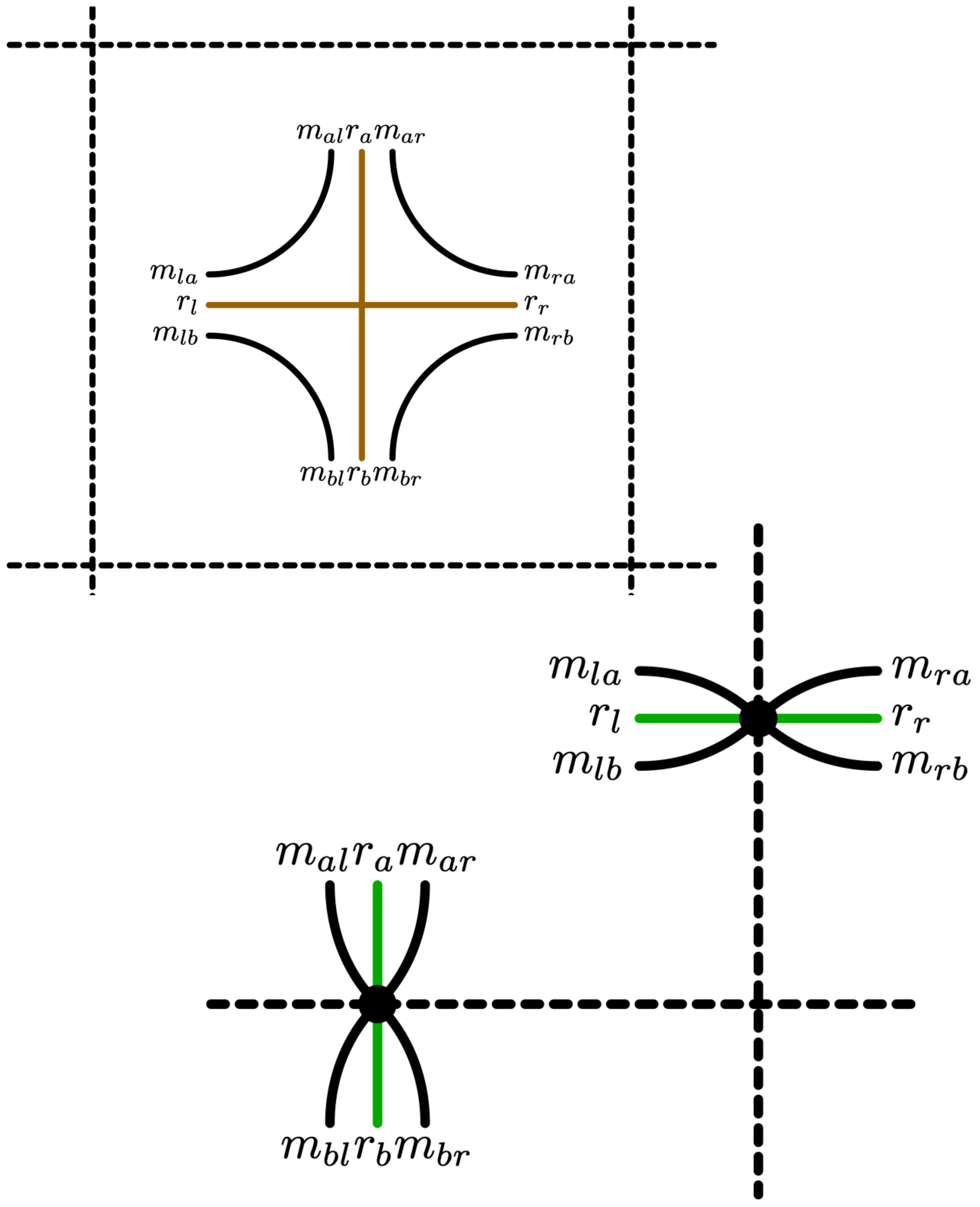


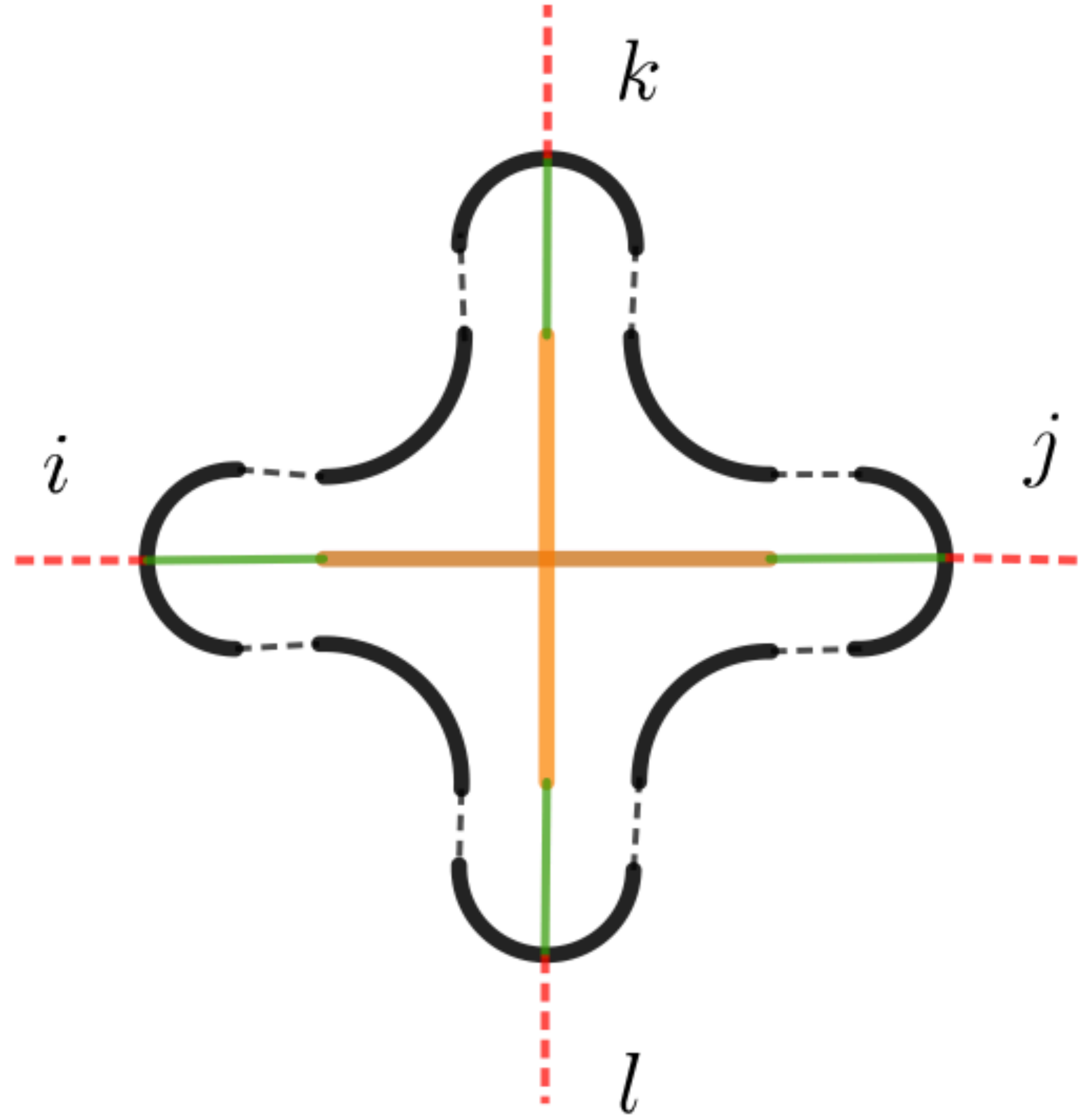
$$A = LL^T$$



Matrix Factorization

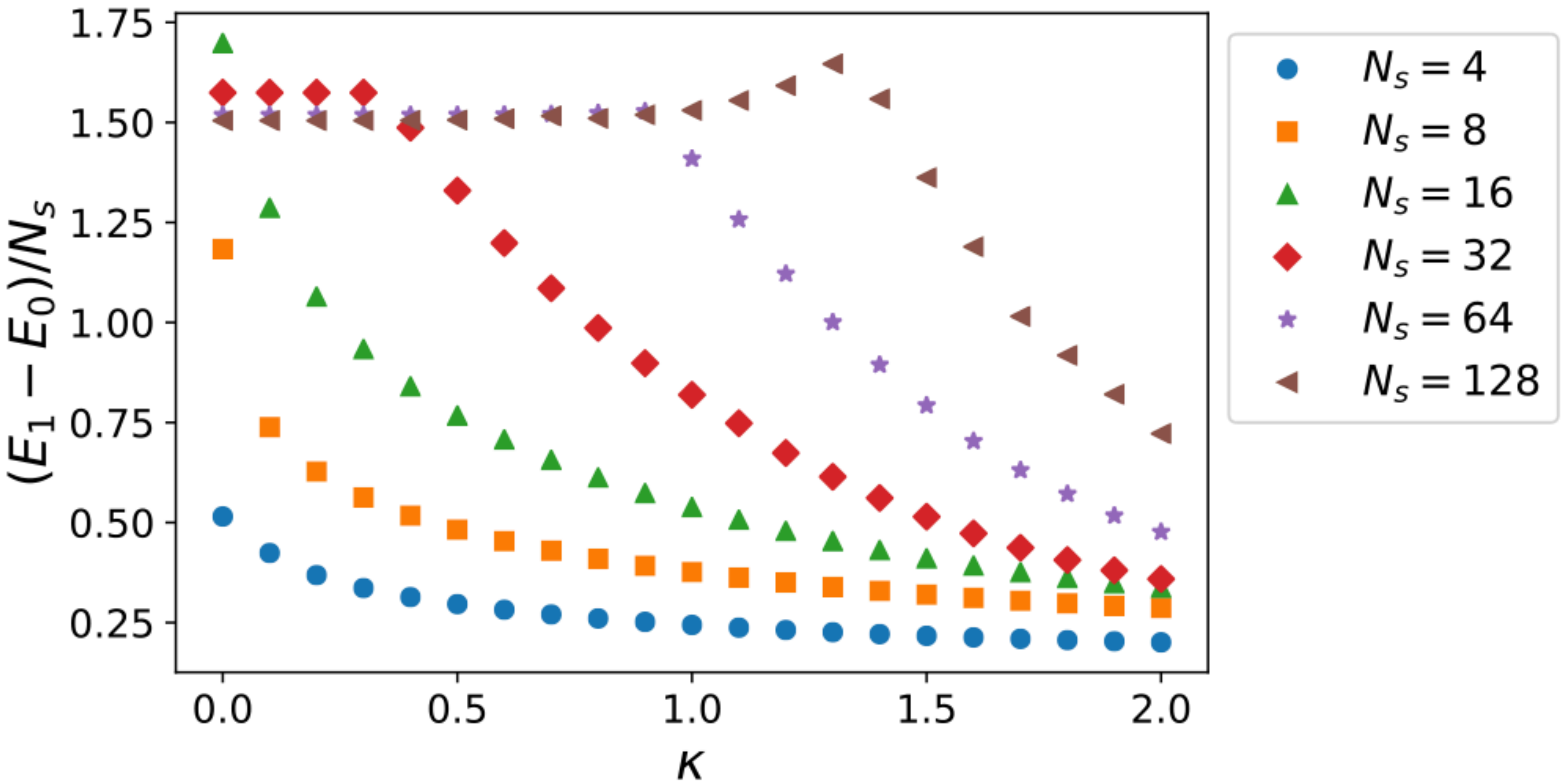
Non-Abelian Higgs Model in 2d





$$Z(\beta, \kappa) = \mathrm{Tr}\left[\prod_x T^{(x)}(\beta, \kappa)\right]$$

Non-Abelian Higgs Model in 2d



Mass gap density as function of κ as continuum limit is taken

Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

Naive discretisation → Fermion Doubling

Staggered fermions — Does not erase the extra fermion copies completely but reduces the number

Since both spinor (not with staggered) and color index is involved with fermions - more tensor legs

- Computationally challenging

Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

$$S_{\mathrm{F}}\left[U\right]=\sum_{n}\left[m\bar{\psi}_{n}\psi_{n}+\sum_{\mu=1}^{2}\frac{\eta_{n,\mu}}{2}\left(\bar{\psi}_{n}U_{n,\mu}\psi_{n+\hat{\mu}}-\bar{\psi}_{n+\hat{\mu}}U^{\dagger}_{n,\mu}\psi_{n}\right)\right].$$

$$\begin{aligned}Z_{\mathrm{F}}\left[U\right]&=\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\prod_{n}e^{-S_{\mathrm{F}}[U]}\\&=\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\prod_{n}\prod_{a=1}^{2}\sum_{s_{n}^{a}=0}^{1}\left(-m\bar{\psi}_{n}^{a}\psi_{n}^{a}\right)^{s_{n}^{a}}\\&\quad\cdot\prod_{a,b=1}^{2}\sum_{x_{n,1}=0}^{1}\left(-\frac{\eta_{n,1}}{2}\bar{\psi}_{n}^{a}U_{n,1}^{ab}\psi_{n+\hat{1}}^{b}\right)^{x_{n,1}^{ab}}\sum_{x_{n,2}=0}^{1}\left(\frac{\eta_{n,1}}{2}\bar{\psi}_{n+\hat{1}}^{a}U_{n,1}^{ba*}\psi_{n}^{b}\right)^{x_{n,2}^{ab}}\\&\quad\cdot\sum_{t_{n,1}=0}^{1}\left(-\frac{\eta_{n,2}}{2}\bar{\psi}_{n}^{a}U_{n,2}^{ab}\psi_{n+\hat{2}}^{b}\right)^{t_{n,1}^{ab}}\sum_{t_{n,2}=0}^{1}\left(\frac{\eta_{n,2}}{2}\bar{\psi}_{n+\hat{2}}^{a}U_{n,2}^{ba*}\psi_{n}^{b}\right)^{t_{n,2}^{ab}}\end{aligned}$$

Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

- $\psi, \bar{\psi}$
- Color index
- Hopping index

We are not even considering the bond dimension from gauge sector as yet

Bond dimension $2^{2\times2\times2} = 256$ with each fermion link

Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

Solution the paper provides is reduced staggered fermions

Place ψ and $\bar{\psi}$ on odd and even sites respectively

$$\psi_n \to (1-\epsilon_n)\psi_n/2$$

$$\bar{\psi}_n \to (1+\epsilon_n)\psi_n/2$$

$$S_{\mathrm{F}}[U] = \sum_n \sum_{\mu=1}^{2} \frac{\eta_{n,\mu}}{2} \psi_n^{\mathrm{T}} \mathcal{U}_{n,\mu} \psi_{n+\hat{\mu}}.$$

Bond dimension reduced to $2^{2\times 2} = 16$

Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

Proceeded with character expansion

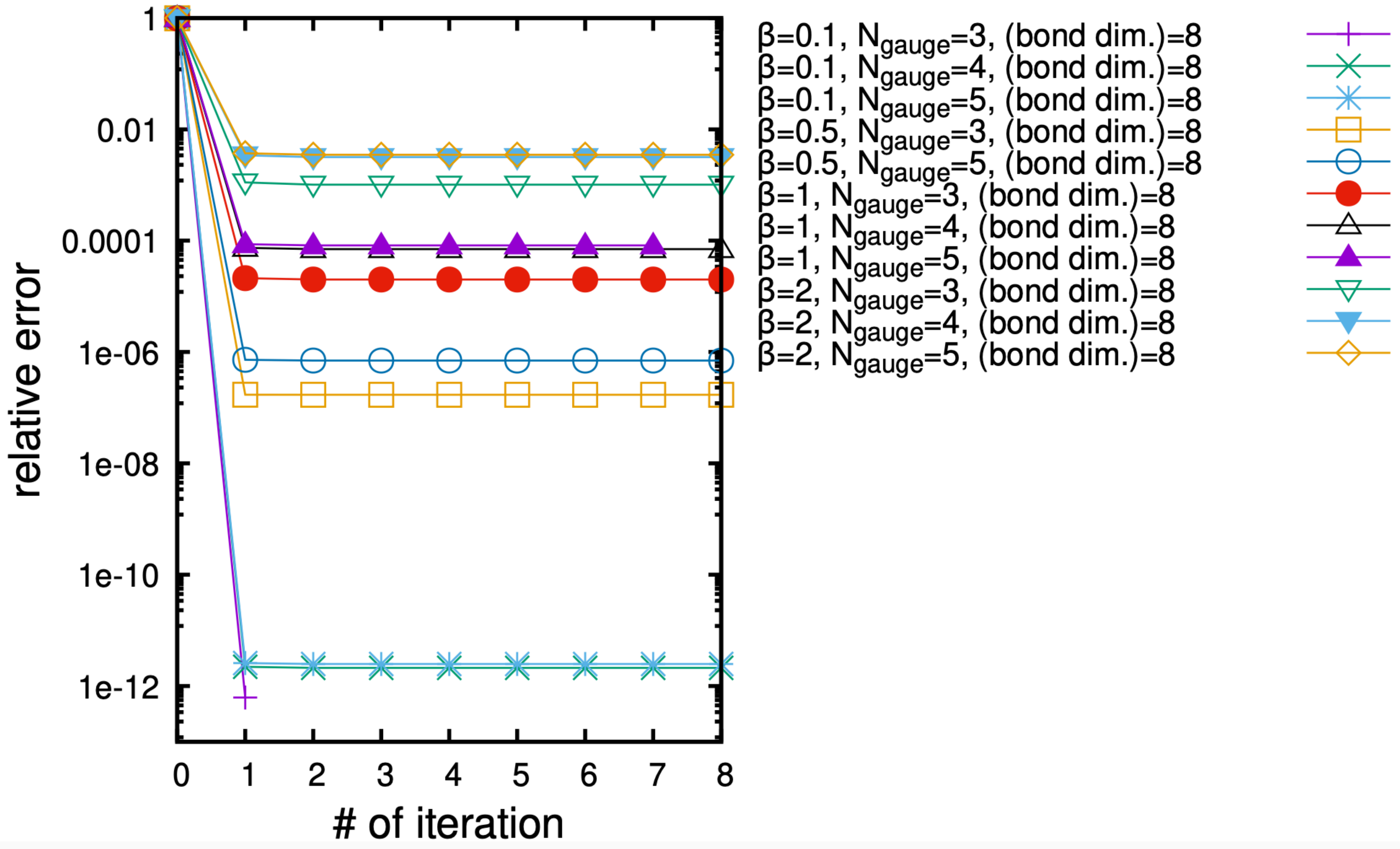
Instead of using HOSVD for plaquette tensor they used HOOI

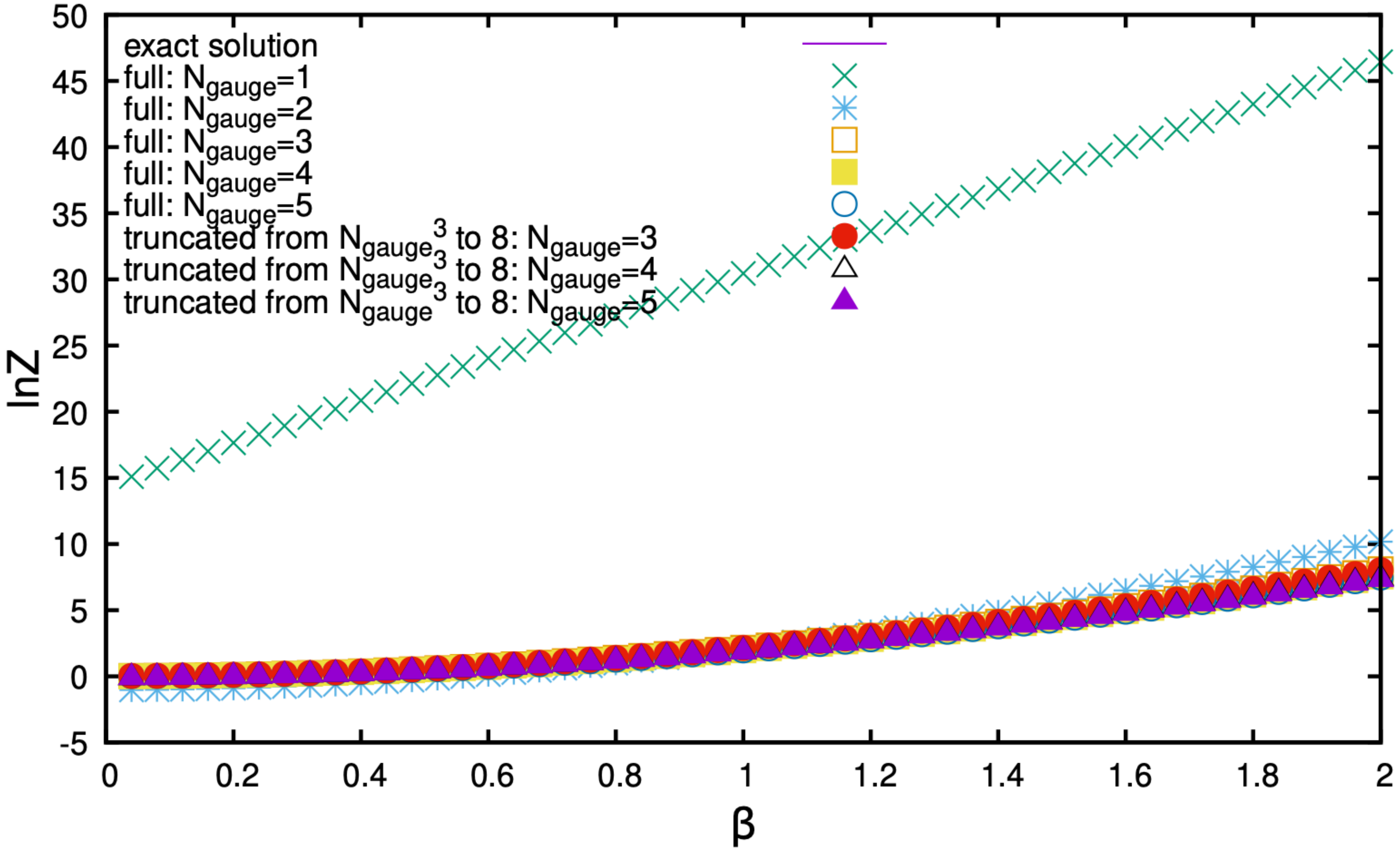
Advantage both in terms of memory and CPU

Process is to obtain a tensor with bond dimension less than original tensor

Asaduzzaman, Catterall, Meurice, Toga, Sakai arXiv:2312.16167 [hep-lat]

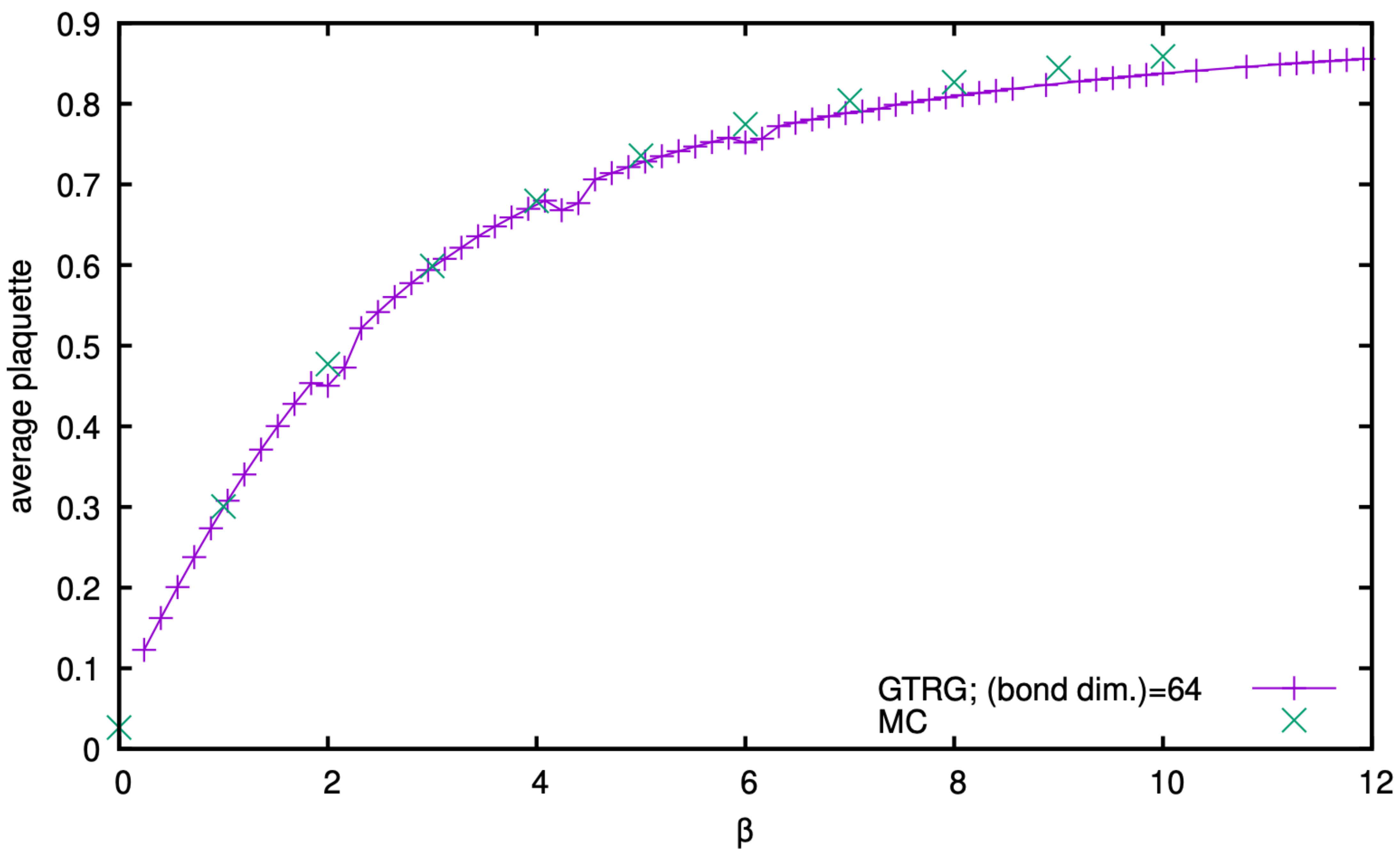
Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions

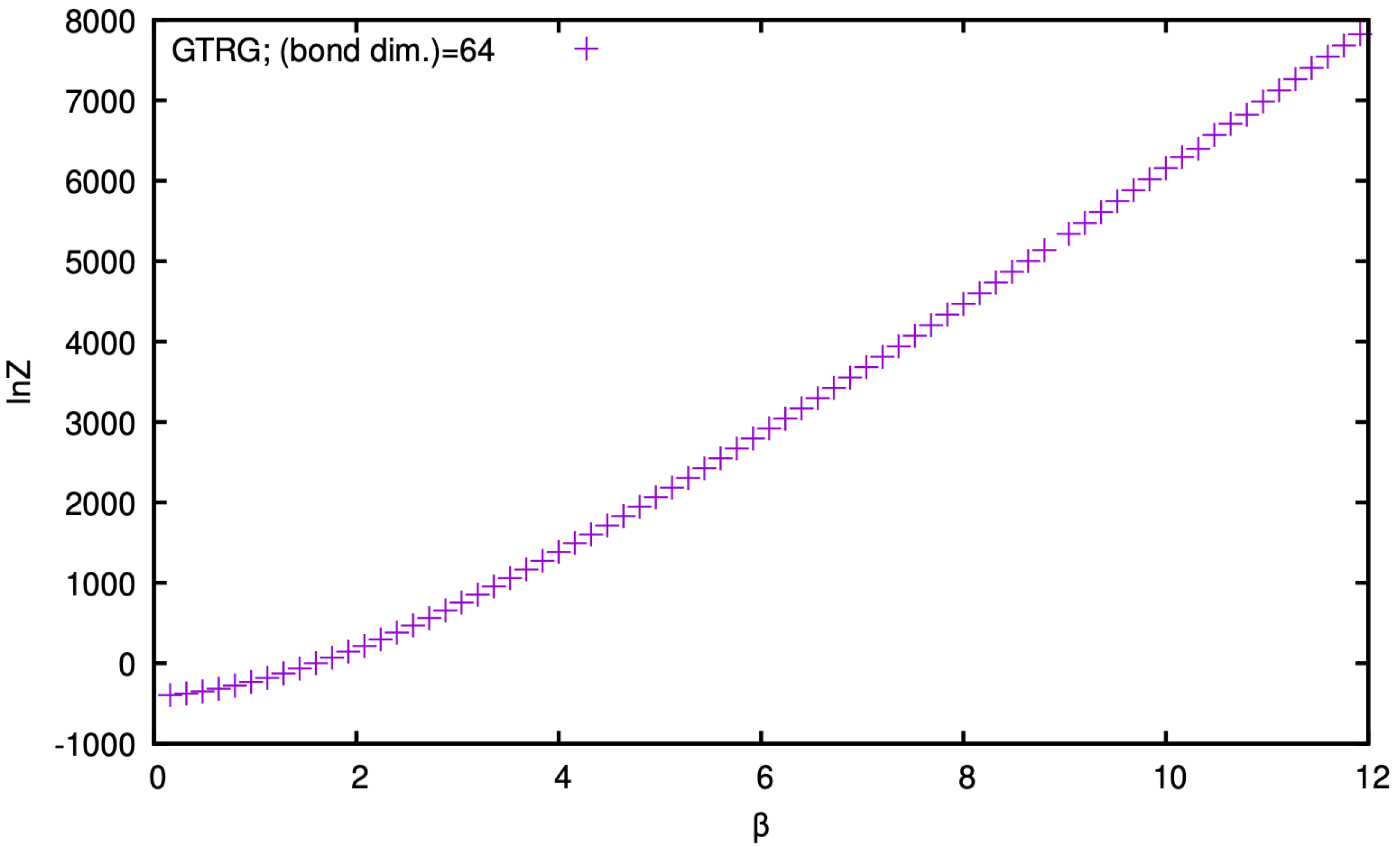




Without fermions

Non-Abelian Gauge Theory Coupled to Reduced Staggered Fermions





With fermions

THANK YOU

Navdeep Singh Dhindsa 15/02/2024