Large Matrices on Lattice and Holography

IMSc, Chennai

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Lattice Why?

Perturbation successful tool for investigating problems in particle physics but it breaks down for **strongly** interacting systems

- Confinement in **QCD**.
- Incorporating non-perturbative effects.
- Phase transitions.
- Beyond the Standard Model and String theory.

Lattice field theory provides a numerical technique to study non-perturbative phenomena by simulating the interactions of particles on a discrete space-time lattice.

Allows the use of first principles calculations

Lattice How?

With the help of the Euclidean path integral, we can understand the dynamics of the theory by regularising it on a space-time lattice.

Real time to Euclidean path integral by Wick rotation, to avoid oscillations in numerical runs.

$$\mathcal{Z} = \int \mathcal{D}\phi \; e^{iS[\phi(x)]/\hbar} \longrightarrow \mathcal{Z} = \int \mathcal{D}\phi \; e^{-S[\phi]}$$

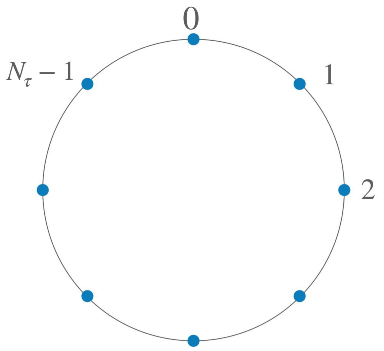
$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\phi \; \mathcal{O}[\phi(x)] \; e^{iS[\phi(x)]/\hbar} \longrightarrow \langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\phi \; \mathcal{O}[\phi] \; e^{-S[\phi]}$$

Example of discretizing fields on a lattice in QM setup

$$\phi(\tau) \to \phi_\tau, \qquad \frac{\partial \phi}{\partial \tau} \to \frac{\phi_{\tau+1} - \phi_\tau}{\mathrm{a}}, \qquad \int_0^\beta \to \mathrm{a} \sum_0^{N_\tau - 1}$$

Lattice

How?



$$\phi(\tau) \to \phi_\tau, \qquad \frac{\partial \phi}{\partial \tau} \to \frac{\phi_{\tau+1} - \phi_\tau}{\mathrm{a}}, \qquad \int_0^\beta \to \mathrm{a} \sum_0^{N_\tau - 1}$$

Fields are simulated on different lattices with the help of **Monte Carlo** method.

Bigger lattices (with fixed size) will help us reach continuum limit.

Increase

Fixed —— $\beta = \mathrm{a} N_\tau$

Decrease

Appropriate set of boundary conditions for different fields

Periodic for Bosons
Anti-periodic for Fermions

$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\phi \, \mathcal{O}[\phi] \, e^{-S[\phi]}$$

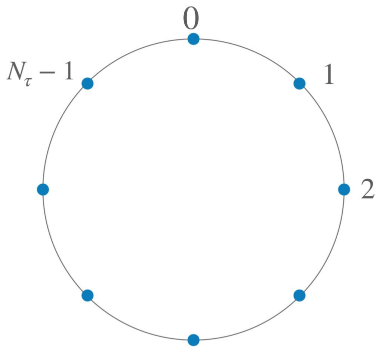
Using Monte Carlo for a large number of steps, we get a Markov chain, which is a sequence of random field configurations

$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\phi^i)$$

Large Matrices

Point like fields → $N \times N$ matrices (which can be many in number depending upon theory)

Connection is also a matrix



Outline

- Holographic motivation for studying theories non-perturbatively
- Lattice setup
- Supersymmetric Yang-Mills and their lattice construction
- Phase structure Bosonic BMN and $\mathcal{N} = (2,2)$ SYM
- Phase structure Conclusions and Future directions

Lattice QCD

On lattice we can study **non-perturbative** aspects of **QCD**

- **Hadron masses**
- **Form factors**
- **Matrix elements**
- **Decay constants**
- **.............**

Gauge/Gravity Duality

Adv. Theor. Math. Phys. ***2*** *(1998) 231-252* *Maldacena*

4d $\mathcal{N} = 4$ SYM dual to Type *IIB* supergravity in decoupling limit

Maximally supersymmetric Yang-Mills (MSYM) theory in p+1 dimensions is dual to Dp-branes in supergravity at low temperatures in large N, strong coupling limit.

PRD ***58*** *(1998) 046004* *Itzhaki et al.*

Gauge/Gravity Duality

Gauge $\leftrightarrow$ Gravity

Strong $\leftrightarrow$ Weak

Hence, if we want to study this conjecture from field theory side, we need a non-perturbative setup.

LATTICE is one such non-perturbative alternative.

Non-perturbative information of String theory with help of AdS/CFT, Matrix Models

- 4d MSYM difficult to simulate using lattice setup as computationally costly.
- This talk will revolve around non-conformal 1d and 2d theories, for which only a handful of lattice studies exist to probe duality.

Supersymmetry

Beautiful and *elegant* way to connect bosons and fermions

$$\mathcal{Q}|\text{Boson}\rangle = |\text{Fermion}\rangle$$

$$\mathcal{Q}|\text{Fermion}\rangle = |\text{Boson}\rangle$$

But experimentally *not observed* and *broken*

Dynamical breaking can only happen because of non-perturbative effects

Standard Model is highly successful

However
- Not UV complete
- Many free parameters
- Hierachy problem
- Dark Matter
- ...

Beyond the SM
- String Theory
- Supersymmetric (SUSY) extension of SM
- Grand Unified Theories

All needs SUSY (in one form or the other)

SUSY on Lattice

SUSY algebra extension of Poincare algebra $\{Q, \overline{Q}\} \sim P_\mu$

P_μ ➜ generates infinitesimal translations ➜ Broken on lattice

Lattice studies of supersymmetric gauge theories

Recent review: *EPJ ST (2022)* *Schaich*

Though SUSY broken on lattice but we can preserve a subset of the algebra

SUSY Yang-Mills theories discretized on lattice using "orbifolding" or "twisting" procedure

*Phys.Rept. **484** (2009) 71-130* *Catterall, Kaplan, Unsal*

SUSY breaking

For supersymmetry broken case, Witten index vanishes.

Vice-versa not generally true.

No SSB

$$|b_{n+1}\rangle = \frac{1}{\sqrt{2E_{n+1}}}\bar{Q}|f_n\rangle, \quad |f_n\rangle = \frac{1}{\sqrt{2E_{n+1}}}Q|b_{n+1}\rangle$$

SSB

$$|b_n\rangle = \frac{1}{\sqrt{2E_n}}\bar{Q}|f_n\rangle, \quad |f_n\rangle = \frac{1}{\sqrt{2E_n}}Q|b_n\rangle$$

Does not vanish ← $$\tilde{\mathcal{Z}} \equiv \mathcal{W} = \mathrm{Tr}\left[(-1)^F e^{-\beta H}\right]$$ → **Vanishes**

Hence AP boundary conditions used throughout runs

$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\phi\, \mathcal{O}[\phi]\, e^{-S[\phi]}$$

Observations using numerical runs unreliable

SUSYQM on Lattice

*Phys. Lett. B **105** (1981) 219-223*

Nielsen, Ninomiya

- Bosonic fields to lattice sites.
- Fermionic fields to lattice sites - Fermionic Doubling

Nielsen-Ninomiya no-go theorem

Not possible to construct lattice fermion action which is:

- Ultra local
- Preserves chiral symmetry
- Has correct continuum limit
- No doublers

Fermions: 4d

- Naive: 16 fermions
- Ginsparg-Wilson: Not ultra local
- Staggered: 4 fermions
- Wilson: 1 fermion, ultra local action but chiral symmetry only recovered in continuum

SUSYQM on Lattice

$$S = \int d\tau \left(-\frac{1}{2}\phi\partial_\tau^2\phi + \overline{\psi}\partial_\tau\psi + \overline{\psi}W''(\phi)\psi + \frac{1}{2}\left[W'(\phi)\right]^2 \right)$$

Still not ready to simulate

$$\mathcal{Z} \;=\; \int \mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{-S_B - S_F}$$

Integrating out fermions

$$\mathcal{Z} \;=\; \int \mathcal{D}\phi \det(M) \; e^{-S_B}$$

- Fermionic matrix size depends upon number of lattice sites
- Computational cost of finding determinant is very high

Hence an alternative is required

PSEUDO-FERMIONS

$$\sqrt{\det(M^T M)} \;=\; \int \mathcal{D}\chi \; e^{-\chi^T (M^T M)^{-1} \chi}$$

Conjugate Gradient Algorithm

*Phys. Lett. B **487** (2000) 349-356* *Catterall, Gregory*

Algorithm

- **RHMC** algorithm
  To deal with fractional powers of fermionic determinant
- **Leapfrog** algorithm
  To evolve the system in simulation time steps
- **Metropolis** test
  To accept/reject the proposed configuration

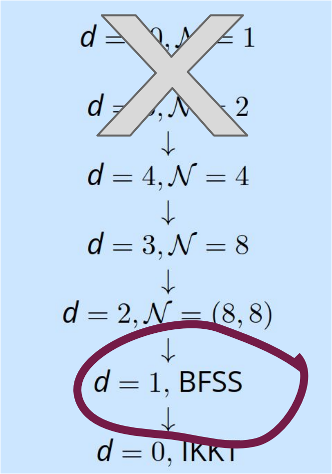
github.com/daschaich/susy

SYM families

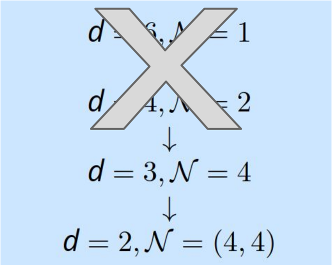
Lower dimensional SYM theories can be constructed by dimensionally reducing higher dimensional

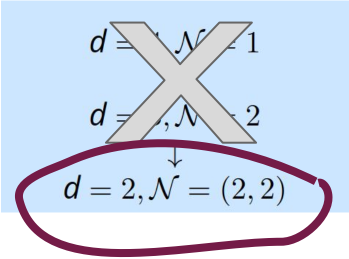
$\mathcal{N} = 1$ SYM theories

16 supersymmetries
Maximal SYM family



8 supersymmetries 4 supersymmetries
Non-Maximal SYM families





Lattice construction using ‘twisting’ requires 2^d supersymmetries

- **MPI** based parallel code.

github.com/daschaich/susy

- Evolved from **MILC** code (which is developed by MIMD Lattice computation collaboration).

- Code is based on distributed memory systems. Can be tested on single-processor workstation or high performance computers.

- Performs **RHMC** simulations of SYM theories in various dimensions.

- Parallelization is between lattice sites, not on matrix degrees.

SUSY on Lattice

Lattice simulations of supersymmetric theories slightly complicated

- Broken SUSY on lattice
- Duality check requires runs at large *N*, computationally expensive
- Flat directions ➔ $[X_i, X_j] = 0$ ➔ but scalar eigenvalues keeps on increasing because of access to continuum branch of the spectra
- Sign problem ➔ Boltzmann factor e^{-S} cannot be used as weight in stochastic process

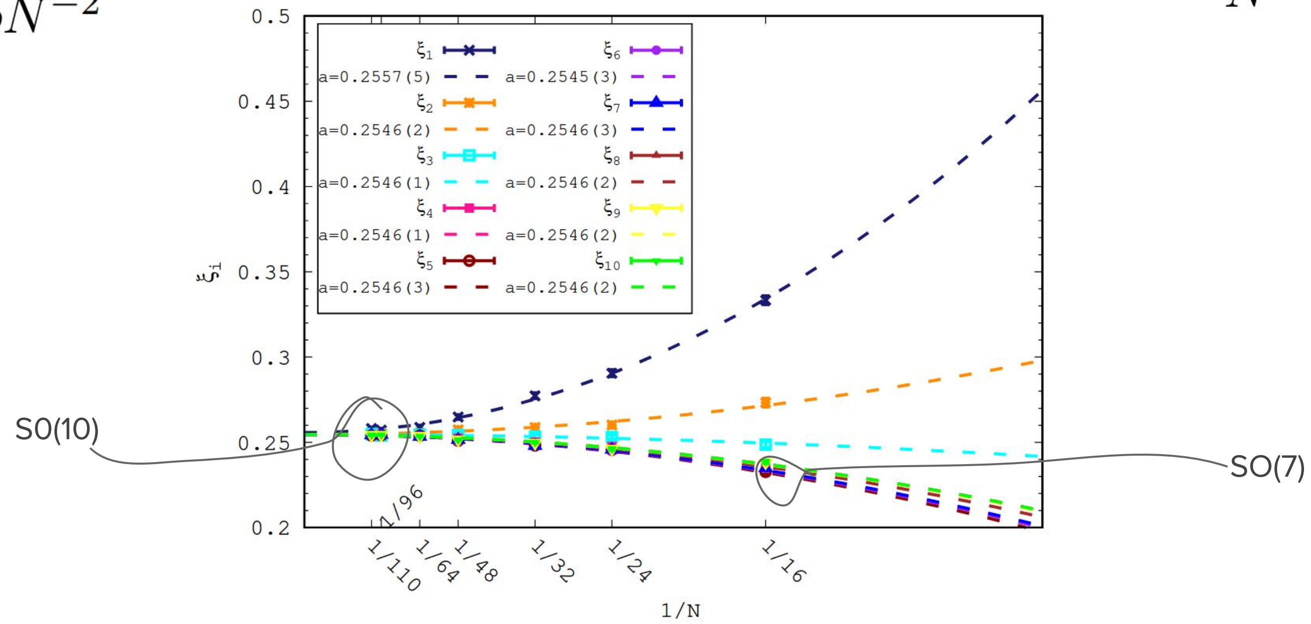
Finite N effects

$$S_{\mathrm{E}} = -\frac{N}{4\lambda}\sum_{i,j}\mathrm{Tr}([X^i, X^j]^2)$$

Will tune eigenvalues of a (10 x 10) matrix constructed out of scalars of bosonic IKKT model

$$I_{ij} = \frac{1}{N}\mathrm{Tr}(X^i X^j)$$

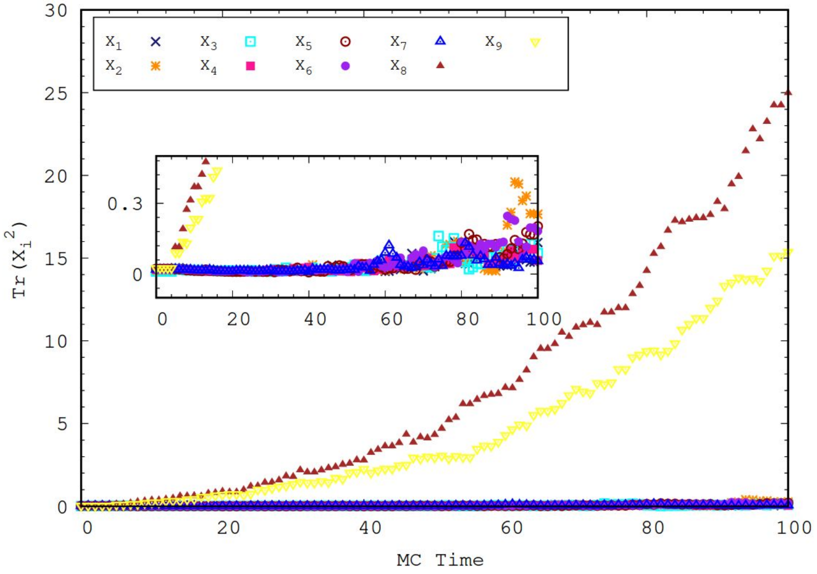
$$a + bN^{-2}$$



Flat directions

BFSS model

Runaway of scalars



This runaway can be controlled by:

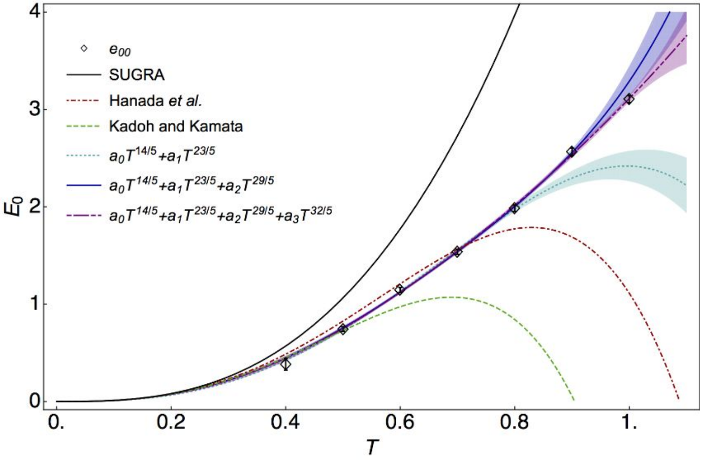
- Adding a deformation term to the action and then fine-tuning it to recover target theory.
- By working with very large ***N***.

Matrix Models

Back to Maximal theories

BFSS Model

$$S_{\rm BFSS} = \frac{N}{4\lambda}\int_0^\beta d\tau\ {\rm Tr}\Big\{ -(D_\tau X_i)^2 - \frac{1}{2}\sum_{i<j}[X_i, X_j]^2 + \Psi_\alpha^T \gamma^\tau_{\alpha\sigma} D_\tau \Psi_\sigma + \Psi_\alpha^T \gamma^i_{\alpha\sigma}[X_i, \Psi_\sigma]\Big\}$$



PRD **58** (2016) 094501   *Hanada et al.*

- SO(9) rotational symmetry

  A recent study using Gaussian expansion shows this symmetry broken like IKKT model
  arXiv:2209.01255   *Brahma, Brandenberger, Laliberte*

- Single deconfined phase in the theory

  A recent study with first results of confined phase
  *JHEP **05** (2022) 096*   *Bergner et al.*

BMN Model

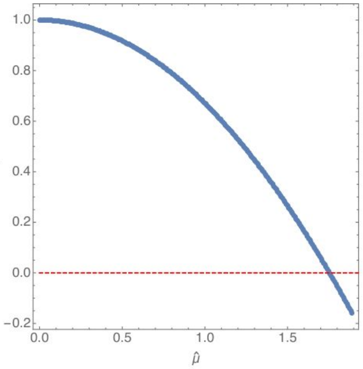
$$S_\mu = -\frac{N}{4\lambda}\int_0^\beta d\tau\ \mathrm{Tr}\left[\left(\frac{\mu}{3}X_I\right)^2 + \left(\frac{\mu}{6}X_A\right)^2 + \frac{\mu}{4}\Psi_\alpha^T\gamma_{\alpha\sigma}^{123}\Psi_\sigma - \frac{\sqrt{2}\mu}{3}\epsilon_{IJK}X_IX_JX_K\right]$$

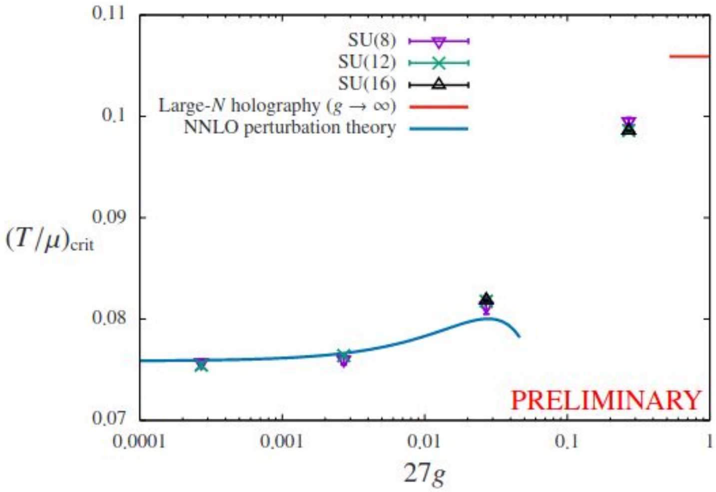
- Mass deformed version of BFSS
- SO(9) explicitly broken into SO(6) X SO(3)
- First order phase transition

Free energy of gravity solution

JHEP **03** (2015) 069

Costa, Greenspan, Penedones, Santos





Numerical simulated results

*PoS **LATTICE21** (2022) 433*

Schaich, Jha, Joseph

Open: Other thermodynamic properties ??

BMN Model

No fermions

➔ Clear deconfinement transition even in BFSS model

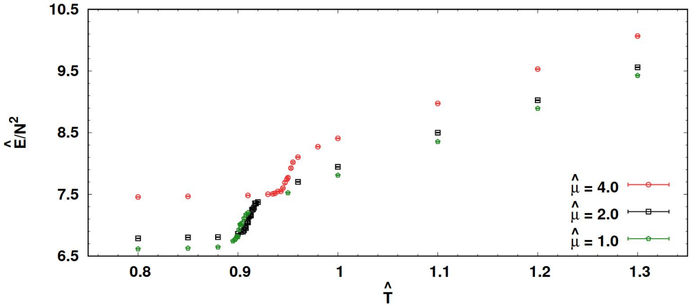
Easier to simulate

➔ Can work with large N setup

$$S_{\text{lat}} = \frac{N}{4\lambda_{\text{lat}}} \sum_{n=0}^{N_\tau - 1} \text{Tr}\Bigg[-(\mathcal{D}_+ X_i)^2 - \frac{1}{2}\sum_{i<j}[X_i, X_j]^2 - \left(\frac{\mu_{\text{lat}}}{3}X_I\right)^2 - \left(\frac{\mu_{\text{lat}}}{6}X_A\right)^2 + \frac{\sqrt{2}\mu_{\text{lat}}}{3}\epsilon_{IJK}X_I X_J X_K \Bigg]$$

Our setup

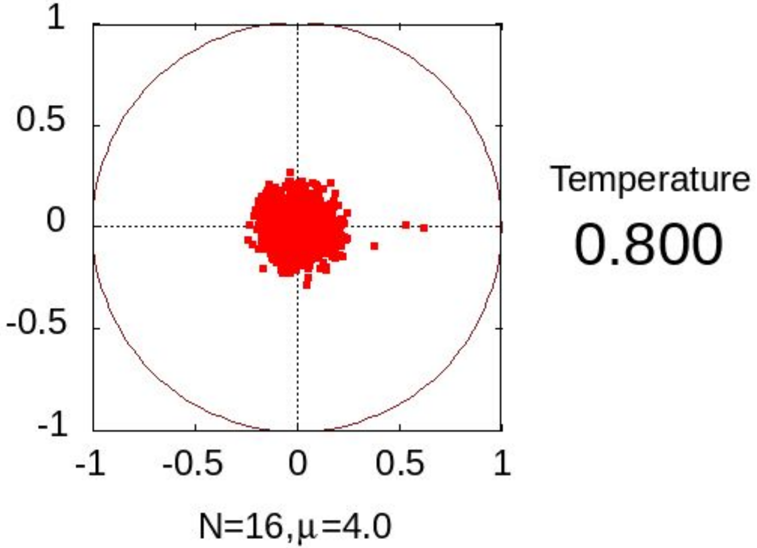
$$\frac{\widehat{E}}{N^2} \equiv \frac{E}{\lambda^{1/3}N^2} = \frac{1}{4N\lambda_{\text{lat}}^{4/3}N_\tau}\left\langle \sum_{n=0}^{N_\tau-1}\text{Tr}\left(-\frac{3}{2}\sum_{i<j}[X_i, X_j]^2 - \frac{2\mu_{\text{lat}}^2}{9}X_I^2 - \frac{\mu_{\text{lat}}^2}{18}X_A^2 + \frac{5\sqrt{2}\mu_{\text{lat}}}{6}\epsilon_{IJK}X^I X^J X^K \right)\right\rangle$$

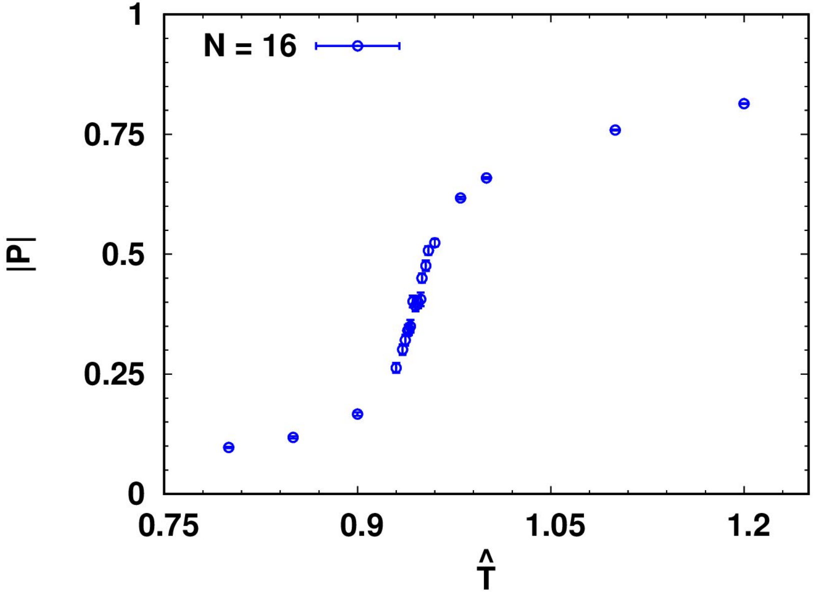


JHEP ***05*** *(2022) 169* **NSD**, Jha, Joseph, Samlodia, Schaich

Polyakov Loop

On lattice : $|P| = \left\langle \frac{1}{N} \left| \mathrm{Tr} \left(\prod_{n=0}^{N_\tau - 1} U(n) \right) \right| \right\rangle$

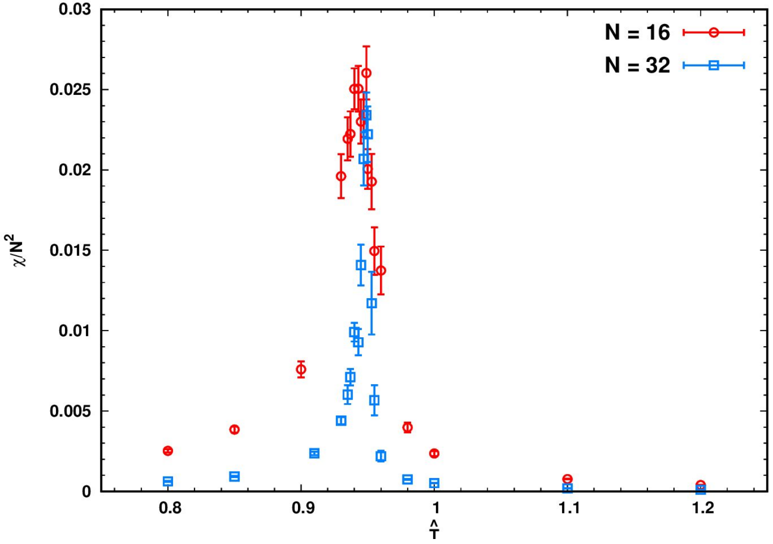




JHEP ***05*** *(2022) 169* **NSD**, Jha, Joseph, Samlodia, Schaich

Transition Order

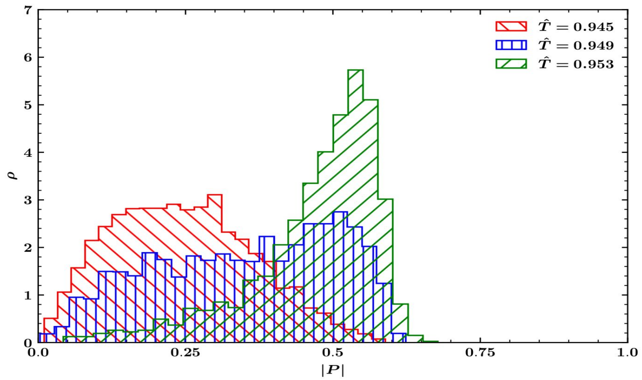
$$\chi \equiv N^2 \left(\langle |P|^2 \rangle - \langle |P| \rangle^2 \right)$$



JHEP ***05*** *(2022) 169* **NSD**, Jha, Joseph, Samlodia, Schaich

- Susceptibility peaks at same height with ***N^2*** normalization
- First order phase transition *PRL* ***113*** *(2014) 091603*

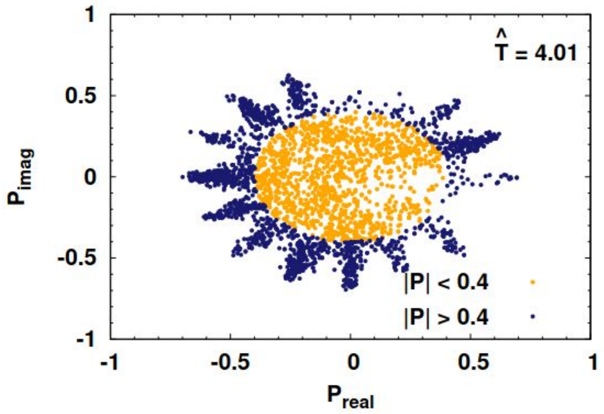
Azuma, Morita, Takeuchi

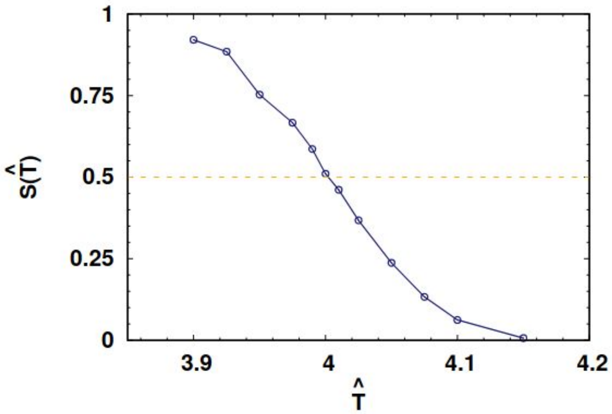


Separatrix Ratio

[PRD **91** (2015) 096002](#)

Francis, Kaczmarek, Laine, Neuhaus, Ohno

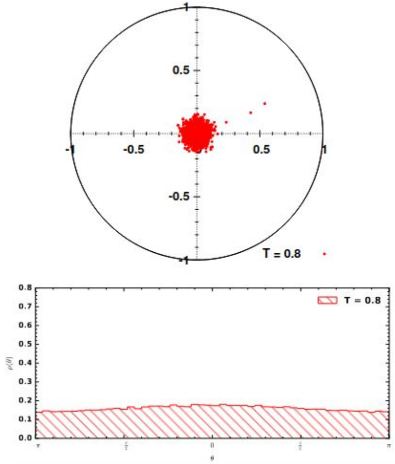




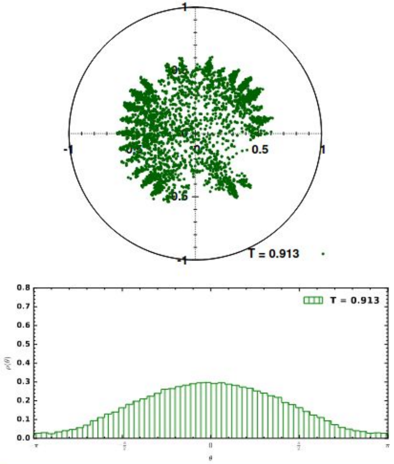
JHEP ***05*** *(2022) 169* **NSD**, Jha, Joseph, Samlodia, Schaich

Different phases

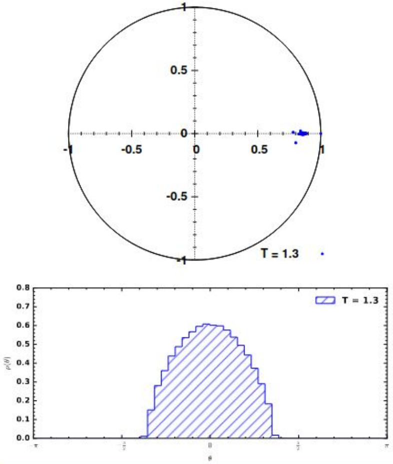
Angular distribution of Polyakov loop eigenvalues



$T = 0.8, \mu_{\text{lat}} = 2.0$
Uniform phase

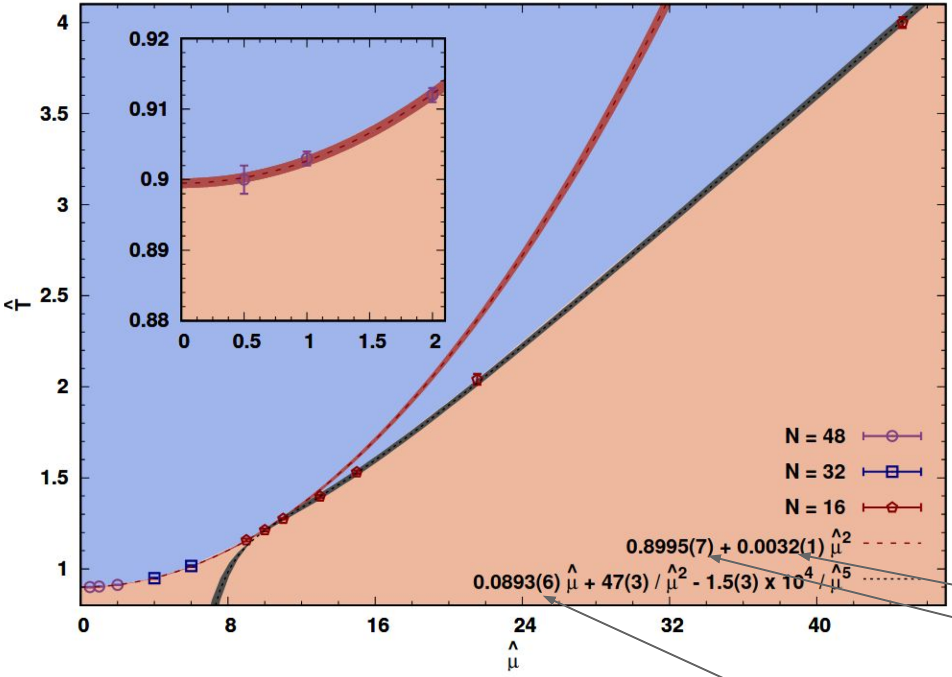


$T = 0.913, \mu_{\text{lat}} = 2.0$
Non-uniform phase



$T = 1.3, \mu_{\text{lat}} = 2.0$
Gapped phase

Phase Diagram



Perturbative calculation valid until $\mu \approx 10$, below it we enter strong coupling regime

First-order phase transition at all couplings

0.00330(2) *JHEP **05** (2022) 096*
0.8846(1) *Bergner et al.*

0.0893 *Adv.Theor.Math.Phys. **8** (2004) 603-696*
Aharony et al.

- Phase diagram smoothly interpolates between bosonic BFSS and gauged Gaussian limit

*JHEP **05** (2022) 169* **NSD**, Jha, Joseph, Samlodia, Schaich

Takeaway Bosonic BMN

- First order phase transition in the model at all values of couplings.
- Perturbative calculations valid upto a certain regime.
- Flat directions do not create any numerical problems, larger N required to get transition points for strong couplings.
- Numerical results smoothly interpolates between bosonic BFSS and gauged Gaussian limit.
- Separatrix method is a viable alternate option to investigate transition point.

JHEP **07** (2013) 101 *Wiseman* ⬅️ About these peculiar powers from SYM

For SYM theory in (1+p) dimensions

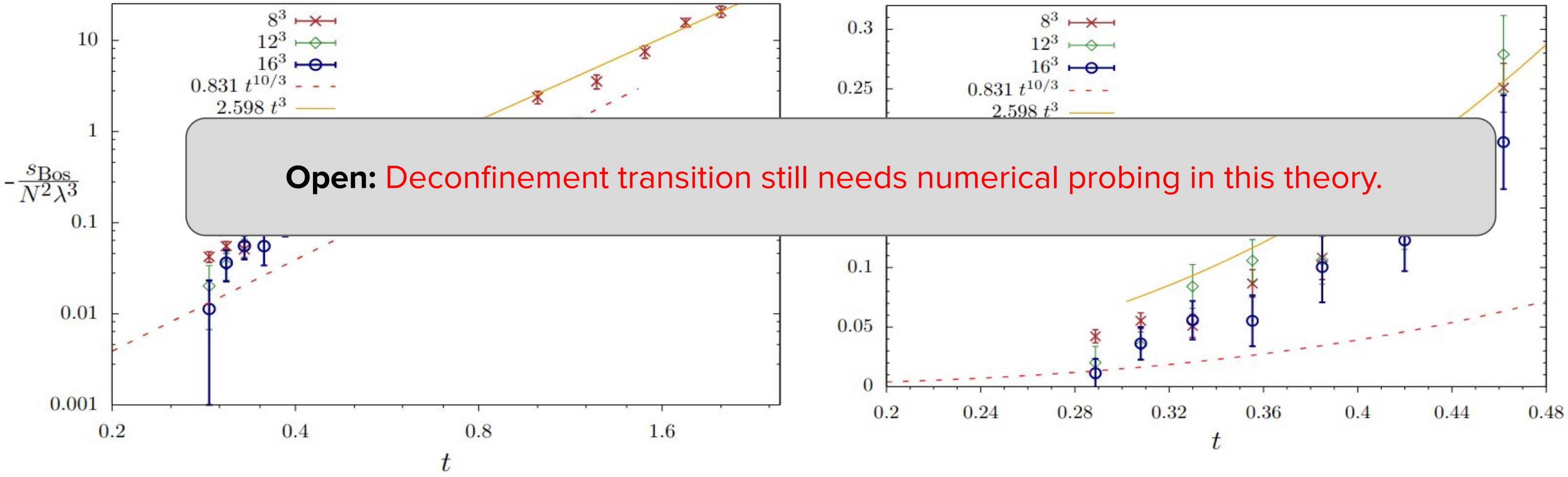
Bosonic action density $\propto t^{p+1}$, $t >> 1$

$\propto t^{(14-2p)/(5-p)}$, $t << 1$

Lattice Results

In conformal case both these cases are equivalent

p = 2



Open: Deconfinement transition still needs numerical probing in this theory.

PRD **102** (2020) 106009 *Catterall, Giedt, Jha, Schaich, Wiseman*

*JHEP **07** (2013) 101* *Wiseman* ⬅️ About these peculiar powers from SYM

For SYM theory in (1+p) dimensions

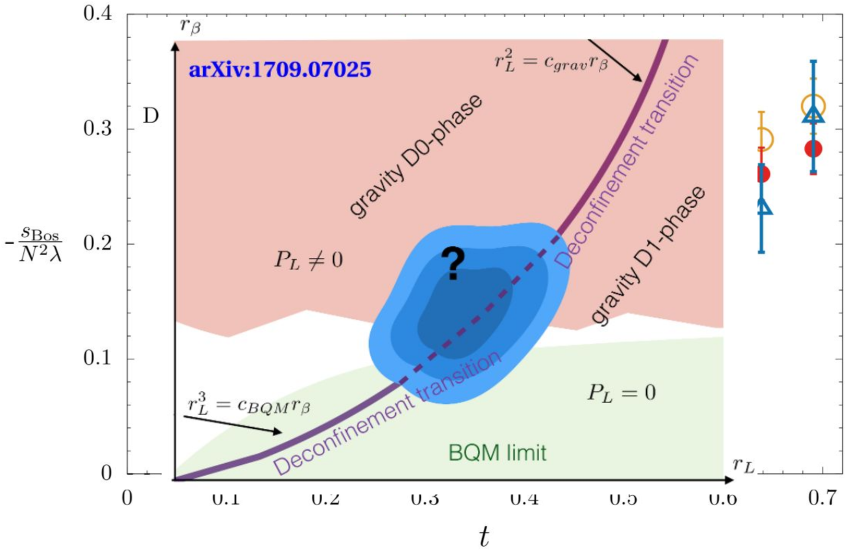
Bosonic action density $\propto t^{p+1}$, $t >> 1$

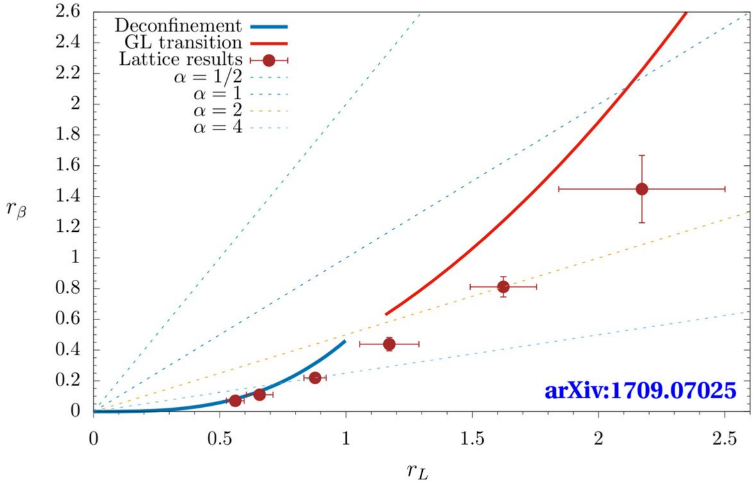
$\propto t^{(14-2p)/(5-p)}$, $t << 1$

Lattice Results

In conformal case both these cases are equivalent

p = 1





*PRD **97** (2018) 086020* *Catterall, Jha, Schaich, Wiseman*

2d $\mathcal{Q}$ = 4 SYM

Regularized on lattice using “**twisting**”

Another alternative is “**orbifolding**”

Phys. Rept. 484 (2009) 71-130

Catterall, Kaplan, Unsal

Global symmetry:

Four-dimensional theory

$SO(4)_E \times U(1)$

Two-dimensional theory

$SO(2)_E \times SO(2)_{R_1} \times U(1)_{R_2}$

- Two possible twists possible as symmetry group contains two SO(2)’s

A $$\mathrm{SO}(2)' = \mathrm{diag}\left(\mathrm{SO}(2)_E \times \mathrm{U}(1)_{R_2}\right)$$

B ✓ $$\mathrm{SO}(2)' = \mathrm{diag}\left(\mathrm{SO}(2)_E \times \mathrm{SO}(2)_{R_1}\right)$$

2d $\mathcal{Q} = 4$ SYM

Regularized on lattice using “**twisting**”

Another alternative is “**orbifolding**”

Phys. Rept. 484 (2009) 71-130

Catterall, Kaplan, Unsal

- Untwisted theory: 4 bosonic d.o.f., 4 fermionic d.o.f., 4 real supercharges
- Fermions, supercharges decomposed to integer spin representation and scalars, gauge fields combine to give complexified field
- Twisted theory: d.o.f. Fermions and complexified gauge field

η, ψ_a, χ_{ab} $\qquad\qquad$ $\mathcal{A}_a$

2d $\mathcal{Q} = 4$ SYM

$$\eta, \psi_a, \chi_{ab}$$

Fermions

- Obtained by dimensionally reducing $\mathcal{N} = 1$ SYM in 4d
- No holographic description

$$S = \frac{N}{4\lambda} \mathcal{Q} \int d^2x \ \mathrm{Tr} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \left[\overline{\mathcal{D}}_a, \mathcal{D}_a \right] - \frac{1}{2} \eta d \right)$$

$$[\mathcal{D}_a, \mathcal{D}_b]$$

$$\partial_a + \mathcal{A}_a$$

$$A_a + iX_a$$

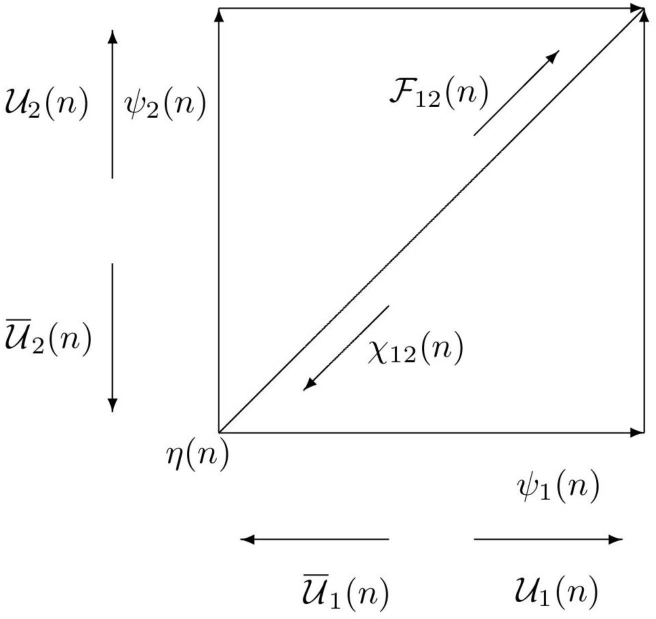
$$\mathcal{Q}\mathcal{A}_a = \psi_a, \quad \mathcal{Q}\overline{\mathcal{A}}_a = 0, \quad \mathcal{Q}\psi_a = 0,$$

$$\mathcal{Q}\chi_{ab} = -\overline{\mathcal{F}}_{ab}, \quad \mathcal{Q}\eta = d, \quad \mathcal{Q}d = 0.$$

After performing $\mathcal{Q}$ variation

2d $\mathcal{Q}$ = 4 SYM

$$S = \frac{N}{4\lambda}\int d^2x \, \mathrm{Tr}\left(-\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} + \frac{1}{2}\left[\overline{\mathcal{D}}_a, \mathcal{D}_a\right]^2 - \chi_{ab}\mathcal{D}_{[a}\,\psi_{\,b]} - \eta\overline{\mathcal{D}}_a\psi_a \right)$$



On Lattice

- Gauge field → Wilson link $\mathscr{A}_a(x)$ → $\mathscr{U}_a(n)$, on links of square lattice
- To preserve SUSY $\psi_a(n)$ lives on same links as bosonic superpartners
- η(n) associated with site
- χ_{ab} (n) lives on diagonal

$$S = \frac{N}{4\lambda_{\mathrm{lat}}}\sum_n \mathrm{Tr}\Bigg[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) + \frac{1}{2}\left(\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n)\right)^2 - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{\,b]}(n) - \eta(n)\overline{\mathcal{D}}_a^{(-)}\psi_a(n)\Bigg],$$

Simulation setup

- To control flat directions

$$S_{\text{total}} = S + \frac{N\mu^2}{4\lambda_{\text{lat}}} \sum_{n,a} \text{Tr}\left(\overline{\mathcal{U}}_a(n)\mathcal{U}_a(n) - \mathbb{I}_N\right)^2$$

- Worked with different mass deformations

$$\mu = \zeta \frac{r_\tau}{N_\tau} = \zeta\sqrt{\lambda}\mathrm{a} = \zeta\sqrt{\lambda_{\text{lat}}}$$

- Different aspect ratio lattices

$$\alpha \equiv \frac{r_x}{r_\tau} = \frac{N_x}{N_\tau}$$

- Different gauge groups, anti-periodic boundary conditions for fermions

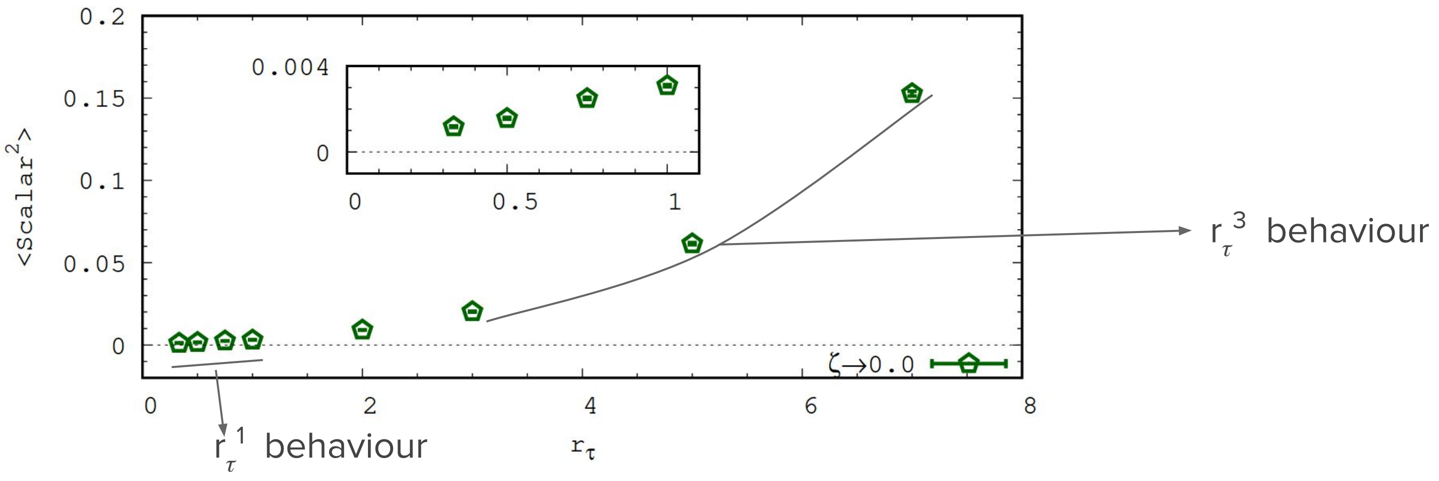
Lattice Results

Scalar^2 → Tr (X^2)
24 x 24 lattice, N =12

*JHEP **07** (2013) 101*

Wiseman

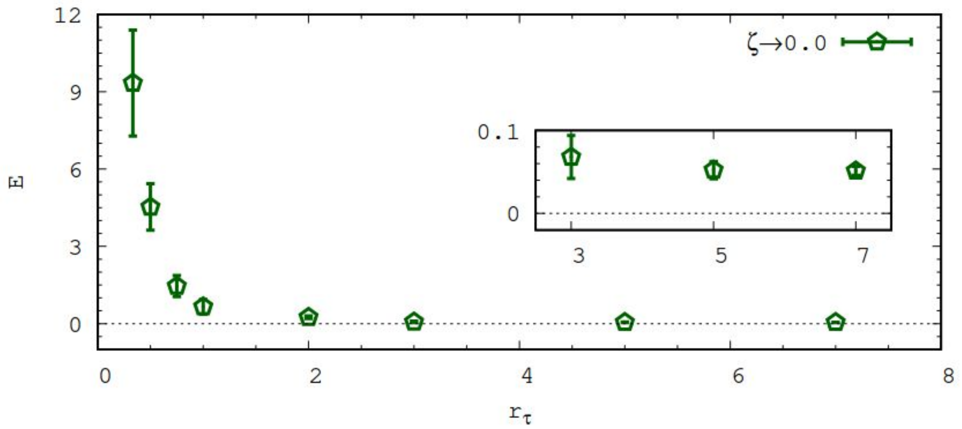
- Behaviour different than maximal cousin
- Existence of bound state at finite temperature



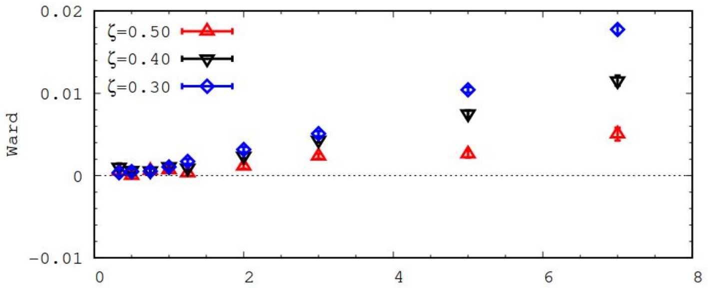
(To appear soon) **NSD**, Jha, Joseph, Schaich

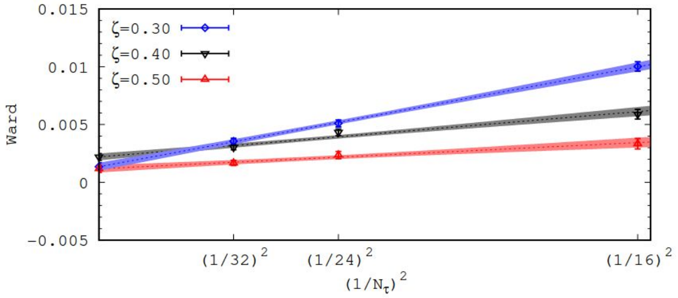
Lattice Results

Preserved SUSY
24 x 24 lattice, N =12



$$\mathcal{Q}\sum_a \left(\eta \mathcal{U}_a \bar{\mathcal{U}}_a\right)$$





$$E = \frac{3}{\lambda_{\text{lat}}}\left(1 - \frac{2}{3N^2} S_B\right)$$

(To appear soon) **NSD**, Jha, Joseph, Schaich

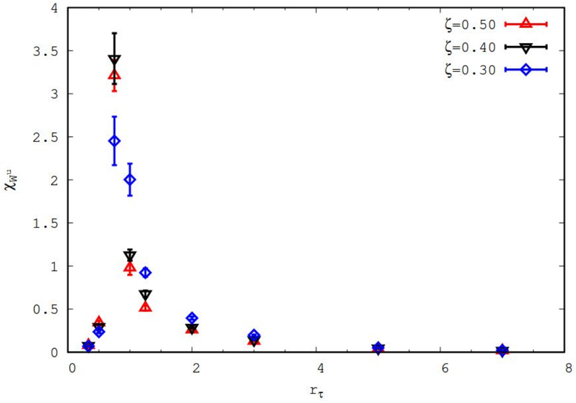
Lattice Results

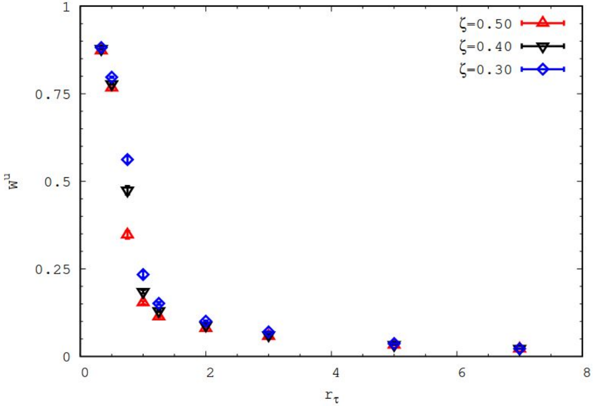
Spatial deconfinement transition

24 x 24 lattice, N =12

MC time history

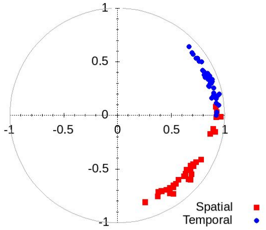
Wilson loop along temporal and spatial direction



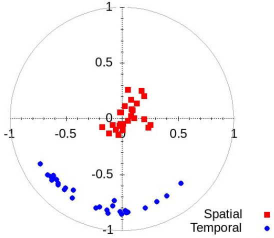


Variance of spatial WL

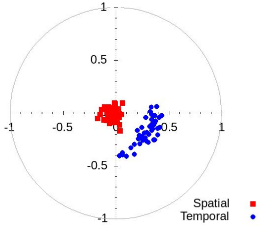
r_τ=0.5,ζ=0.3



r_τ=1.0,ζ=0.3



r_τ=3.0,ζ=0.3

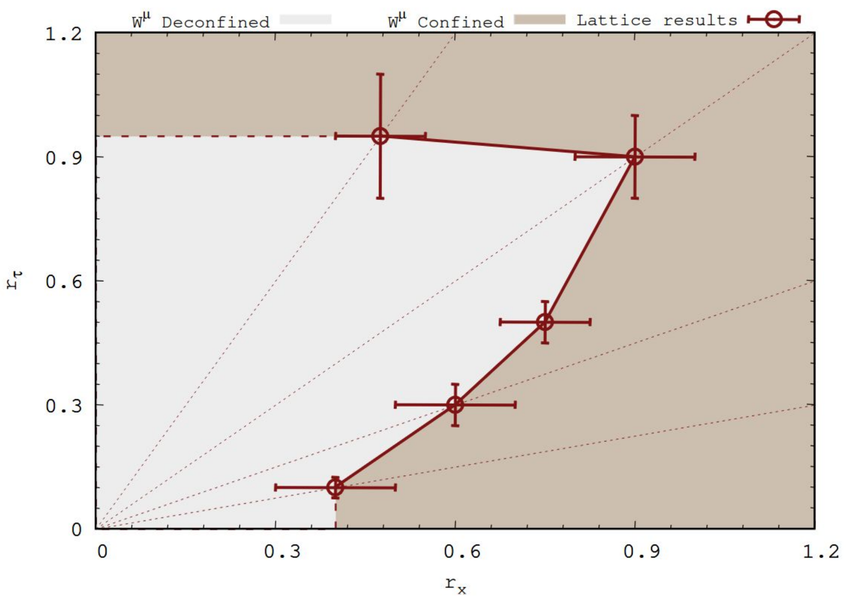


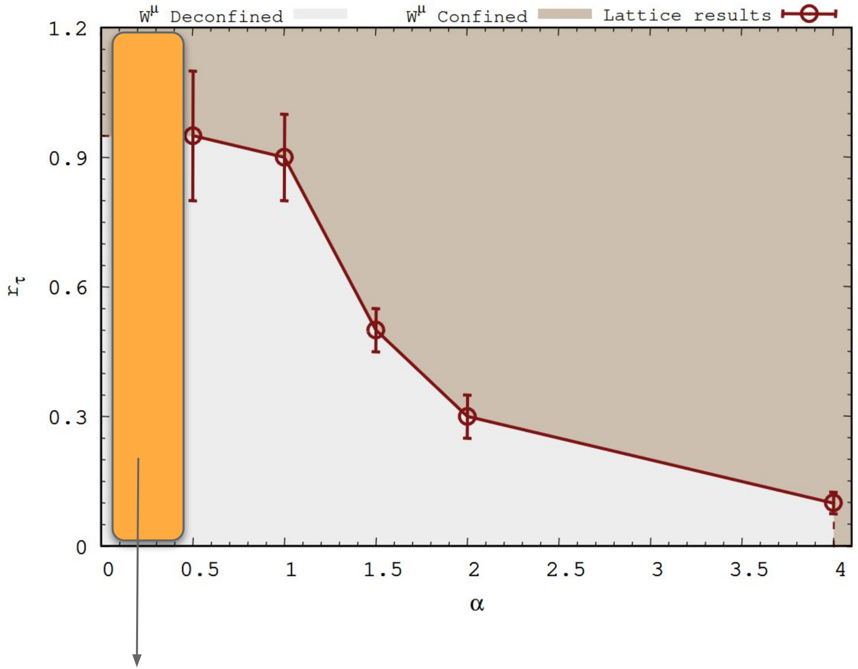
(To appear soon) **NSD**, Jha, Joseph, Schaich

Lattice Results

Phase diagram

Different aspect ratio **α**, N =12





Problematic regime in numerical simulations

(To appear soon) **NSD**, Jha, Joseph, Schaich

Takeaway 2d $\mathcal{Q} = 4$ SYM

- Scalars show bound state behaviour
- Spatial deconfinement transition, but only limited to weak coupling regime
- Thermodynamics different than maximal counterpart
- More analysis required to probe if it admits holographic description : **Open**

Numerical Bootstrap

➔ To derive the spectrum of the theory by checking the positivity of some of the observables.

- Taking the help of loop equations to connect various orders of observables.

$$\mathcal{M} = \begin{bmatrix} \left\langle O_0^\dagger O_0 \right\rangle & \left\langle O_0^\dagger O_1 \right\rangle & \cdots & \left\langle O_0^\dagger O_K \right\rangle \\ \left\langle O_1^\dagger O_0 \right\rangle & \left\langle O_1^\dagger O_1 \right\rangle & \cdots & \left\langle O_1^\dagger O_K \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle O_K^\dagger O_0 \right\rangle & \left\langle O_K^\dagger O_1 \right\rangle & \cdots & \left\langle O_K^\dagger O_K \right\rangle \end{bmatrix} \geq 0$$

Numerical Bootstrap

$$V = m\frac{X^2}{2} + g\frac{X^4}{4}$$

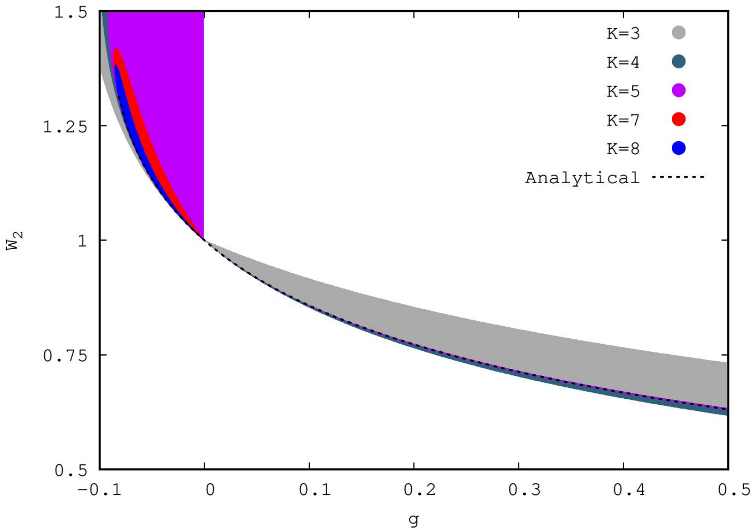
$$mW^n + gW^{n+2} = \sum_{j=0}^{n-2} W^j W^{n-2-j}$$

$$\left\langle \frac{1}{N}\mathrm{Tr}\left(X^2\right)\right\rangle = \frac{(12g+m^2)^{1.5} - 18mg - m^3}{54g^2}$$

$$\mathcal{M} = \begin{bmatrix} \langle X^0\rangle & \langle X^1\rangle & \langle X^2\rangle & \cdots & \langle X^K\rangle \\ \langle X^1\rangle & \langle X^2\rangle & \langle X^3\rangle & \cdots & \langle X^{K+1}\rangle \\ \vdots & \vdots & \ddots & \vdots & \\ \langle X^K\rangle & \langle X^{K+1}\rangle & \langle X^{K+2}\rangle & \cdots & \langle X^{2K}\rangle \end{bmatrix} \geq 0$$

Plot with m = 1

- This plot generated in less than 1 minute.
- But gets complicated as number of matrices increase

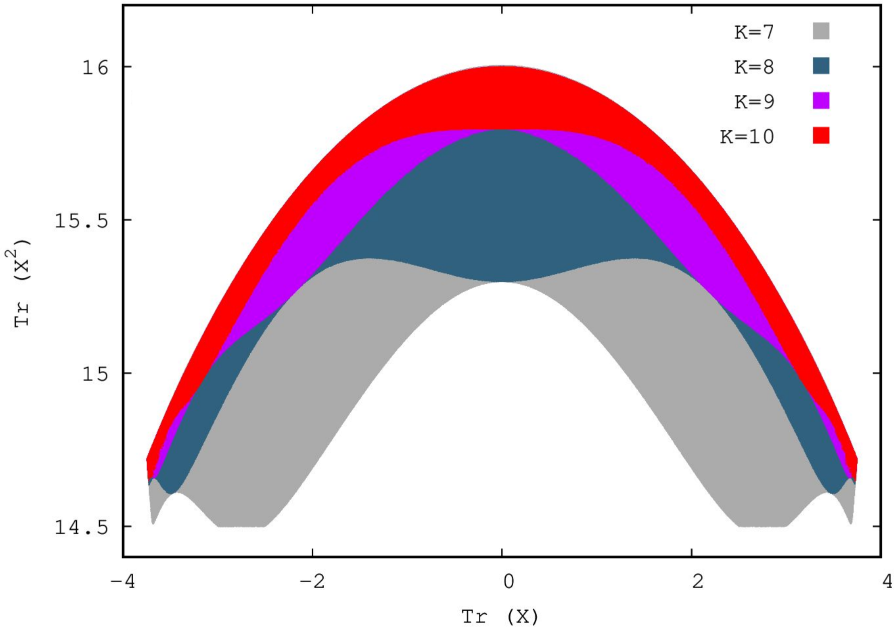


Numerical Bootstrap

$$V = m\frac{X^2}{2} + g\frac{X^4}{4}$$

➔ Also useful when we have curve of solutions

Plot with m = -1, g = 1/16



Can we improve Monte Carlo to sample all the vacua in large *N* limit?

Future Directions

- Numerical tools beyond Monte Carlo, especially for lower dimensional models
  - Numerical bootstrap is a viable option to investigate Matrix Models *JHEP **06** (2020) 090* *Lin*

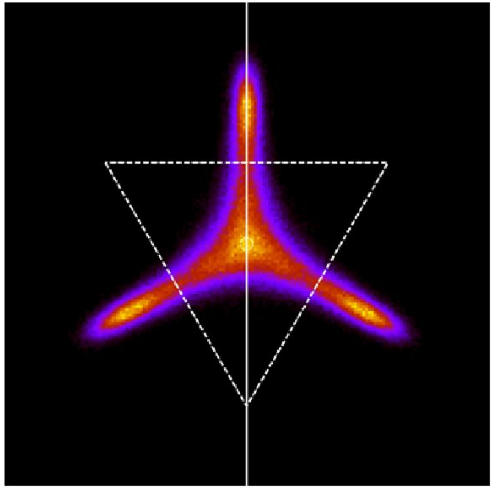
- Numerically investigating non-gauge/gravity *JHEP **04** (2018) 084* *Maldacena, Milekhin*
  - Recent numerical results *JHEP **08** (2022) 178* *Pateloudis et al.*

- Continue exploring non-maximal and maximal supersymmetric theories

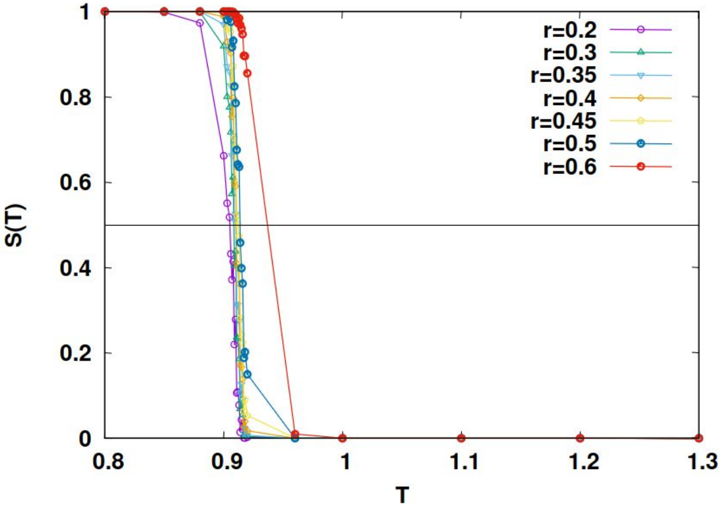
- Improving Monte Carlo Method

THANK YOU

Separatrix

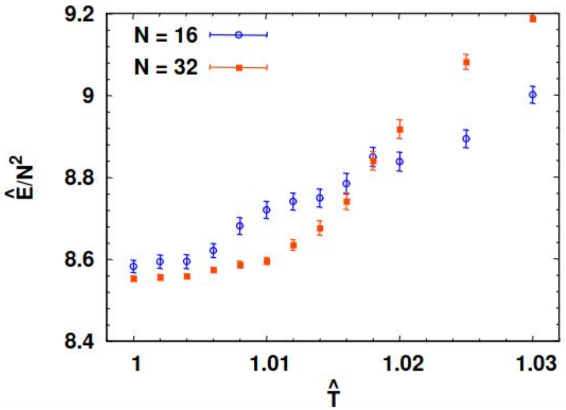


PRD 91 (2015) 096002

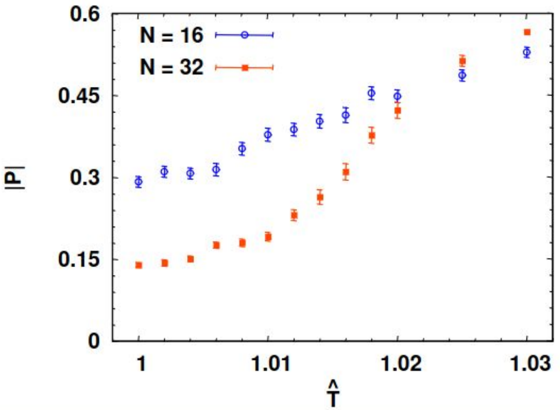


Separatrix ratio vs r $N = 32, \hat{\mu} = 2$

BBMN Results



Energy $\hat{\mu} = 6$



Polyakov Loop $\hat{\mu} = 6$

First order transition

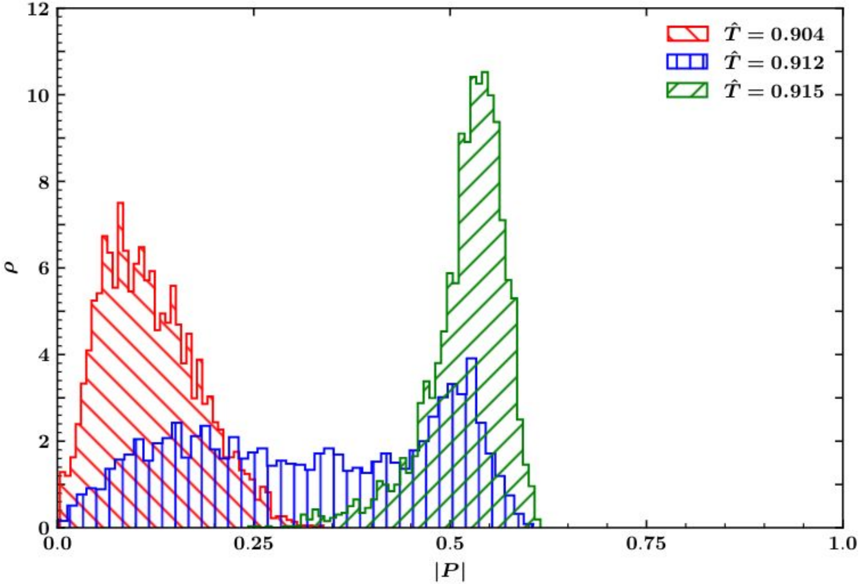


FIGURE 4.12: Polyakov loop magnitude distribution at three different temperatures for $\widehat{\mu} = 2.0$ with $N = 48$. A two-peak structure appears to develop more clearly as compared with lower N values.

AP BC Fermions

Thermal green function

$$G_B(x, y, \tau_1, \tau_2) = Z^{-1} Tr\left[e^{-\beta K} T\left[\hat{\phi}(x, \tau_1)\hat{\phi}(y, \tau_2)\right]\right]$$

using step fn. with $\tau_1 = \tau$, $\tau_2 = 0$ and cyclic property of trace

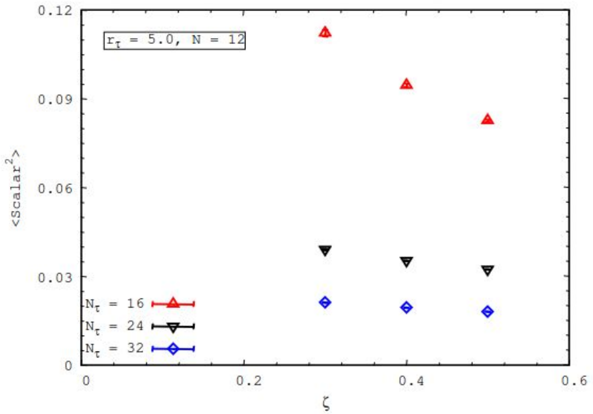
$$G_B(x, y, \tau, 0) = Z^{-1} Tr\left[\hat{\phi}(y, 0)e^{-\beta K}\hat{\phi}(x, \tau)\right]$$

$$G_B(x, y, \tau, 0) = Z^{-1} Tr\left[e^{-\beta K}e^{+\beta K}\hat{\phi}(y, 0)e^{-\beta K}\hat{\phi}(x, \tau)\right]$$

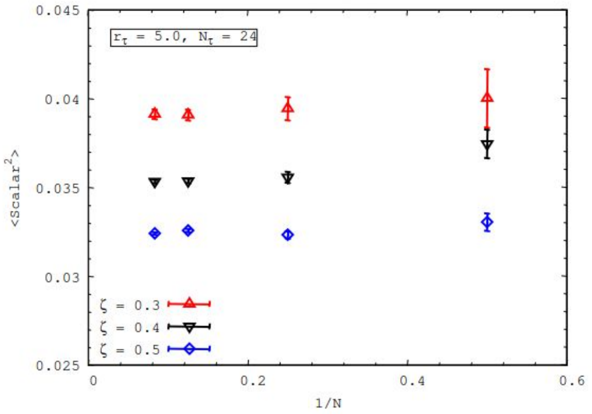
$$G_B(x, y, \tau, 0) = Z^{-1} Tr\left[e^{-\beta K}\hat{\phi}(y, \beta)\hat{\phi}(x, \tau)\right]$$

If ϕ's are bosons last two interchanged gives $\phi(y, \beta) = \phi(y, 0)$, if ϕ's are fermions (say ψ) last two interchanged gives extra -ve sign $\psi(y, \beta) = -\psi(y, 0)$, hence APBC for fermions

Bound state 2d

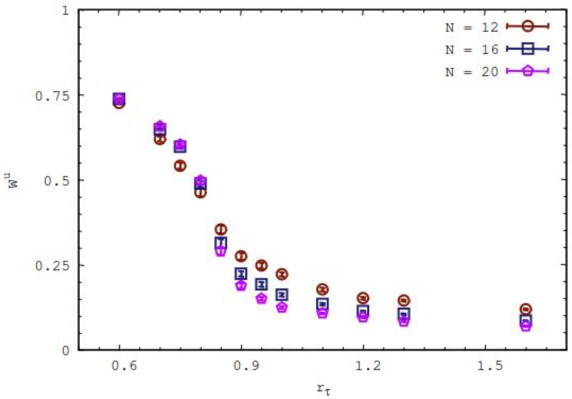


Bound state vs lattice size

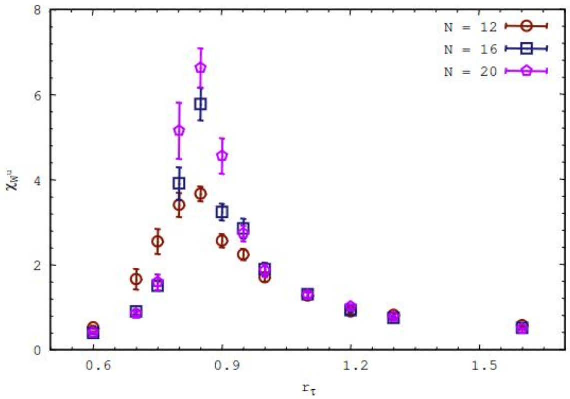


Bound state vs gauge group

Transition order 2d

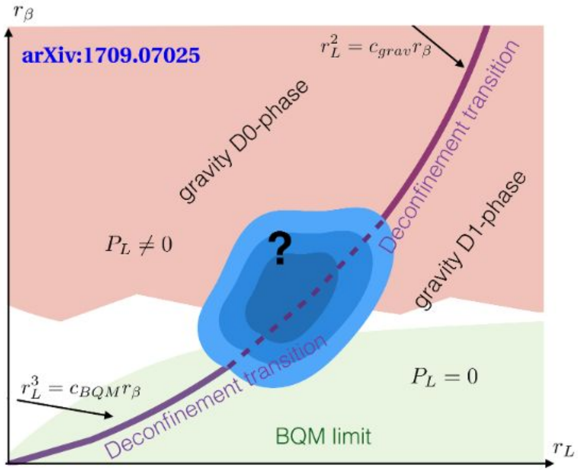


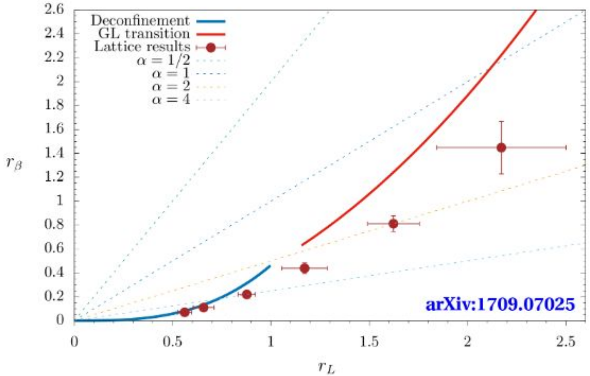
Wilson loop dependence on *N*



χ vs *N* hints second order phase transition

Maximal theory 2d





$\mathcal{Q} = 16$

Fermion doubling

Dirac propagator free theory:

$$S = \frac{m - ia^{-1}\sum_\mu \gamma^\mu \sin(p^\mu a)}{m^2 + a^{-2}\sum_\mu \sin(p^\mu a)^2}$$

For low momenta pole at $p^\mu a = (am, 0, 0, 0)$

But fifteen additional poles at $p^\mu a = (am, 0, 0, 0) + \pi^\mu$

As $sin(p^\mu a)$ has two poles in range $p^\mu = [-\pi/a, \pi/a]$