Ph.D. Thesis Defense

20.03.2023

NON-PERTURBATIVE STUDIES OF NON-CONFORMAL FIELD THEORIES

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PH18086

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Lattice Why?

Perturbation successful tool for investigating problems in particle physics but it breaks down for **strongly** interacting systems

- Confinement in **QCD**.
- Incorporating non-perturbative effects.
- Phase transitions.
- Beyond the Standard Model and String theory.

Lattice field theory provides a numerical technique to study non-perturbative phenomena by simulating the interactions of particles on a discrete space-time lattice.

Allows the use of first principles calculations

Lattice How?

With the help of the Euclidean path integral, we can understand the dynamics of the theory by regularising it on a space-time lattice.

Real time to Euclidean path integral by Wick rotation, to avoid oscillations in numerical runs.

$$\mathcal{Z} = \int \mathcal{D}\phi \; e^{iS[\phi(x)]/\hbar} \longrightarrow \mathcal{Z} = \int \mathcal{D}\phi \; e^{-S[\phi]}$$

$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\phi \; \mathcal{O}[\phi(x)] \; e^{iS[\phi(x)]/\hbar} \longrightarrow \langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\phi \; \mathcal{O}[\phi] \; e^{-S[\phi]}$$

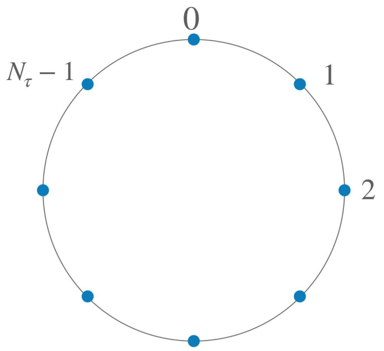
Example of discretizing fields on a lattice in QM setup

$$\phi(\tau) \rightarrow \phi_\tau, \qquad \frac{\partial \phi}{\partial \tau} \rightarrow \frac{\phi_{\tau+1} - \phi_\tau}{\mathrm{a}}, \qquad \int_0^\beta \rightarrow \mathrm{a} \sum_0^{N_\tau - 1}$$

Lattice

How?

$$\phi(\tau) \to \phi_\tau, \qquad \frac{\partial \phi}{\partial \tau} \to \frac{\phi_{\tau+1} - \phi_\tau}{\mathrm{a}}, \qquad \int_0^\beta \to \mathrm{a} \sum_0^{N_\tau - 1}$$



Fields are simulated on different lattices with the help of **Monte Carlo** method.

Bigger lattices (with fixed size) will help us reach continuum limit.

Increase

Fixed $\beta = \mathrm{a} N_\tau$

Decrease

Appropriate set of boundary conditions for different fields

Periodic for Bosons
Anti-periodic for Fermions

Using Monte Carlo for a large number of steps, we get a Markov chain, which is a sequence of random field configurations

$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\phi \, \mathcal{O}[\phi] \, e^{-S[\phi]}$$

$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\phi^i)$$

Outline

- Holographic motivation for studying theories non-perturbatively
- Supersymmetric Quantum Mechanics
- Supersymmetric Yang-Mills and their lattice construction
- Phase structure Bosonic BMN
- Phase structure $\mathcal{N} = (2,2)$ SYM
- Conclusions and Future directions

Gauge/Gravity Duality

Adv. Theor. Math. Phys. ***2*** *(1998) 231-252* *Maldacena*

4d $\mathcal{N} = 4$ SYM dual to Type *IIB* supergravity in decoupling limit

Maximally supersymmetric Yang-Mills (MSYM) theory in p+1 dimensions is dual to Dp-branes in supergravity at low temperatures in large N, strong coupling limit.

PRD ***58*** *(1998) 046004* *Itzhaki et al.*

Gauge/Gravity Duality

Gauge $\leftrightarrow$ Gravity

Strong $\leftrightarrow$ Weak

Hence, if we want to study this conjecture from field theory side, we need a non-perturbative setup.

LATTICE is one such non-perturbative alternative.

Non-perturbative information of String theory with help of AdS/CFT, Matrix Models

- 4d MSYM difficult to simulate using lattice setup as computationally costly.
- This talk will revolve around non-conformal 1d and 2d theories, for which only a handful of lattice studies exist to probe duality.

Supersymmetry

Beautiful and *elegant* way to connect bosons and fermions

$$\mathcal{Q}|\text{Boson}\rangle = |\text{Fermion}\rangle$$

$$\mathcal{Q}|\text{Fermion}\rangle = |\text{Boson}\rangle$$

But experimentally *not observed* and *broken*

Dynamical breaking can only happen because of non-perturbative effects

Standard Model is highly successful

However
- Not UV complete
- Many free parameters
- Hierachy problem
- Dark Matter
- ...

Beyond the SM
- String Theory
- Supersymmetric (SUSY) extension of SM
- Grand Unified Theories

All needs SUSY (in one form or the other)

SUSY on Lattice

SUSY algebra extension of Poincare algebra $\{Q, \overline{Q}\} \sim P_\mu$

P_μ ➜ generates infinitesimal translations ➜ Broken on lattice

Lattice studies of supersymmetric gauge theories

Recent review: *EPJ ST (2022)* *Schaich*

Though SUSY broken on lattice but we can preserve a subset of the algebra

SUSY Yang-Mills theories discretized on lattice using “orbifolding” or “twisting” procedure

*Phys.Rept. **484** (2009) 71-130* *Catterall, Kaplan, Unsal*


Eur. Phys. J. Plus (2022) 137:1155
https://doi.org/10.1140/epjp/s13360-022-03389-w

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Probing non-perturbative supersymmetry breaking through lattice path integrals

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Abstract We investigate non-perturbative supersymmetry breaking in various models of quantum mechanics, including an interesting class of PT-invariant models, using lattice path integrals. These theories are discretized on a temporal Euclidean lattice with anti-periodic boundary conditions. Hybrid Monte Carlo algorithm is used to update the field configurations to their equilibrium values. We used the Ward identities, expectation value of the action, and the expectation value of the first derivative of the superpotential as tools for probing supersymmetry breaking.

1 Introduction

We can use supersymmetric quantum mechanics as a testbed to illustrate several properties of systems containing bosons and fermions. Since Witten's seminal work [1], the idea of non-perturbative supersymmetry (SUSY) breaking has been investigated extensively in the literature. These investigations range from studying the properties of supersymmetric quantum mechanics to supersymmetric gauge theories in various spacetime dimensions [2–11]. In this work, we investigate non-perturbative SUSY breaking in various quantum mechanics models by regularizing them on a Euclidean lattice. Supersymmetric quantum mechanics models have been the subject of thorough investigations in the context of various physical systems over the past few decades (see Ref. [12] for a review). For example, the model with a quartic superpotential (supersymmetric anharmonic oscillator) has been simulated on the lattice, by several groups, over the past ten years or so, with great success (see Refs. [13–19]). In this work, we explore supersymmetric quantum mechanics with various types of superpotentials, including the interesting class of PT-invariant potentials. After verifying the existing simulation results in the literature on supersymmetric anharmonic oscillator, with the help of a lattice regularized action, and an efficient simulation algorithm, we use the same setup to probe SUSY breaking in models with three different superpotentials. They include a degree-five potential, a shape-invariant potential of Scarf I type, and a certain type of PT-invariant potential. Although the

Before investigating SYM theories: Let's investigate a simpler example of SUSY Quantum Mechanics

Based on

Eur. Phys. J. Plus ***137**, 1155 (2022)* ***NSD,*** *Joseph*

- A testbed to understand supersymmetry on lattice.
- Supersymmetry broken/preserved checked for different superpotentials.

SUSYQM

Nucl. Phys. B **188** (1981) 513 Witten

$$S = \int_0^\beta d\tau \left(-\frac{1}{2}\phi\partial_\tau^2\phi + \overline{\psi}\partial\right.$$

$$Q\phi = \overline{\psi}, \quad Q\psi = -\partial_\tau\phi + W', \quad Q\overline{\psi} = 0,$$

Supercharges act on different fields as

Action invariant under two supercharges

$$\overline{Q}\phi = -\psi, \quad \overline{Q}\psi = 0, \quad \overline{Q}\,\overline{\psi} = \partial_\tau\phi + W'.$$

- Integrating out auxiliary field ‘B’

$$S = \int d\tau \left(-\frac{1}{2}\phi\partial_\tau^2\phi + \overline{\psi}\partial_\tau\psi + \overline{\psi}W''(\phi)\psi + \frac{1}{2}\left[W'(\phi)\right]^2\right)$$

SUSYQM

For supersymmetry broken case, Witten index vanishes.

Vice-versa not generally true.

No SSB

$$|b_{n+1}\rangle = \frac{1}{\sqrt{2E_{n+1}}}\bar{Q}|f_n\rangle, \quad |f_n\rangle = \frac{1}{\sqrt{2E_{n+1}}}Q|b_{n+1}\rangle$$

SSB

$$|b_n\rangle = \frac{1}{\sqrt{2E_n}}\bar{Q}|f_n\rangle, \quad |f_n\rangle = \frac{1}{\sqrt{2E_n}}Q|b_n\rangle$$

Does not vanish

Vanishes

$$\tilde{\mathcal{Z}} \equiv \mathcal{W} = \mathrm{Tr}\left[(-1)^F e^{-\beta H}\right]$$

Hence AP boundary conditions used throughout runs

$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \mathcal{D}\phi \, \mathcal{O}[\phi] \, e^{-S[\phi]}$$

Observations using numerical runs unreliable

SUSYQM on Lattice

*Phys. Lett. B **105** (1981) 219-223*

Nielsen, Ninomiya

- Bosonic fields to lattice sites.
- Fermionic fields to lattice sites - Fermionic Doubling

Nielsen-Ninomiya no-go theorem

Not possible to construct lattice fermion action which is:

- Ultra local
- Preserves chiral symmetry
- Has correct continuum limit
- No doublers

We use **Wilson** fermions in our setup

$$K_{ij} = m\delta_{ij} - \frac{1}{2}\left(\delta_{i,j+1} + \delta_{i,j-1} - 2\delta_{ij}\right)$$

Fermions: 4d

- Naive: 16 fermions
- Ginsparg-Wilson: Not ultra local
- Staggered: 4 fermions
- Wilson: 1 fermion, ultra local action but chiral symmetry only recovered in continuum

SUSYQM on Lattice

$$S = \int d\tau \left(-\frac{1}{2}\phi\partial_\tau^2\phi + \overline{\psi}\partial_\tau\psi + \overline{\psi}W''(\phi)\psi + \frac{1}{2}\left[W'(\phi)\right]^2 \right)$$

Still not ready to simulate

$$\mathcal{Z} \;=\; \int \mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{-S_B - S_F}$$

Integrating out fermions

$$\mathcal{Z} \;=\; \int \mathcal{D}\phi \,\det(M)\, e^{-S_B}$$

- Fermionic matrix size depends upon number of lattice sites
- Computational cost of finding determinant is very high

Hence an alternative is required

PSEUDO-FERMIONS

$$\sqrt{\det(M^T M)} \;=\; \int \mathcal{D}\chi \; e^{-\chi^T (M^T M)^{-1}\chi}$$

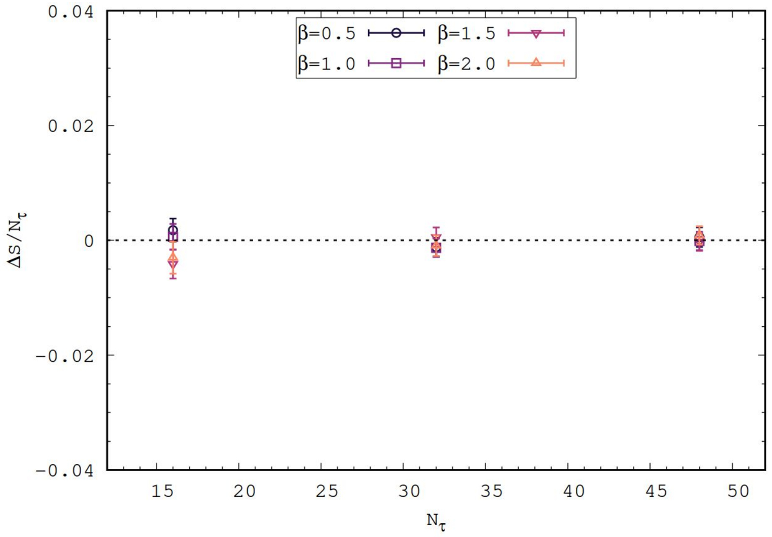
Conjugate Gradient Algorithm

[*Phys. Lett. B* ***487*** *(2000) 349-356*](#) *Catterall, Gregory*

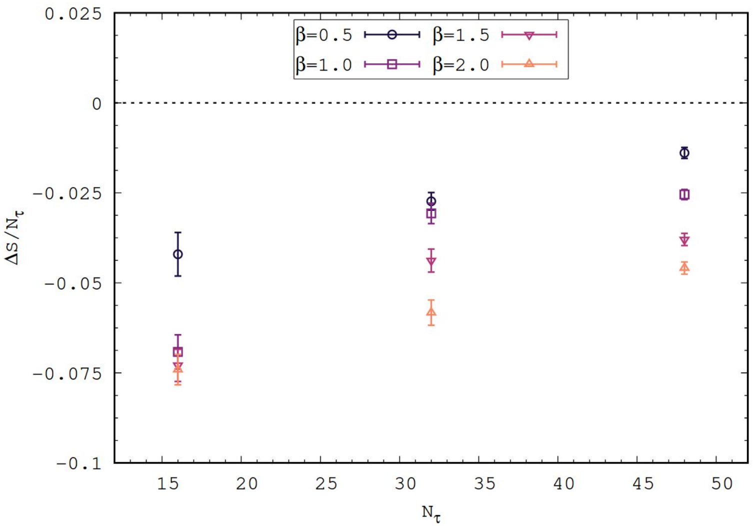
SUSYQM Results

$$\Delta S \equiv \langle S \rangle_{\text{exact}} - \langle S \rangle = N_\tau - \langle S \rangle$$

If the observable is zero then SUSY preserved



$$W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{4}g\phi^4$$

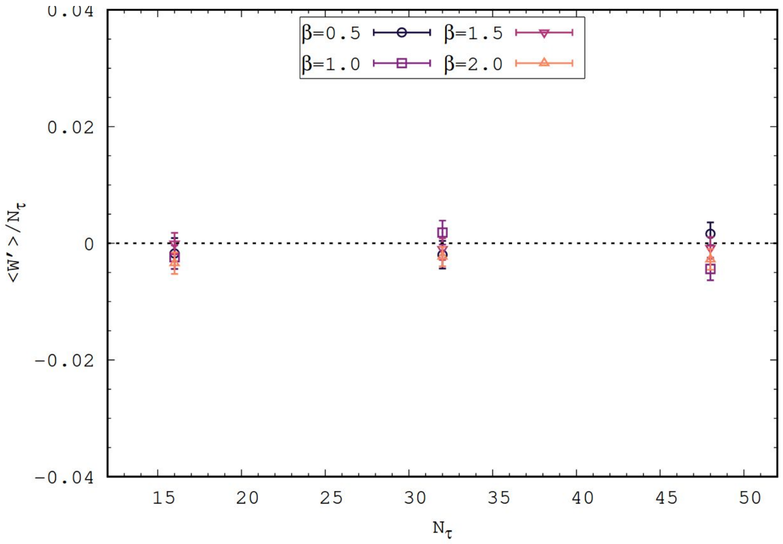


$$W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{5}g\phi^5$$

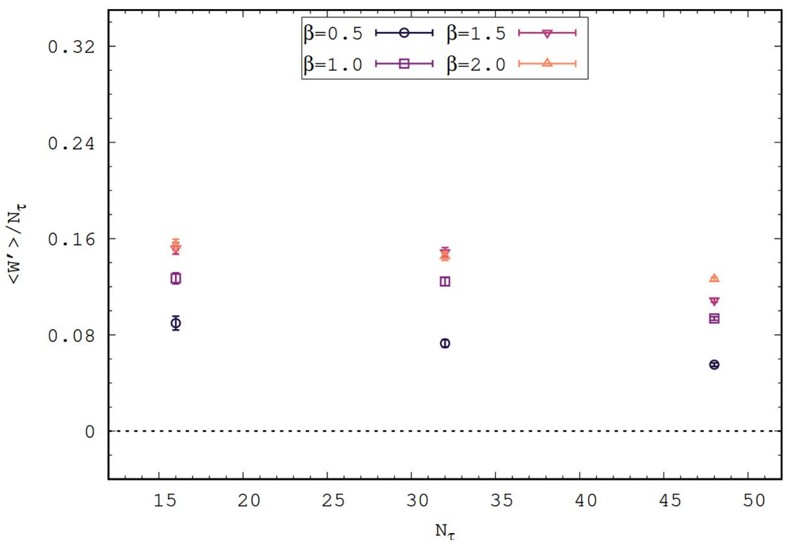
SUSYQM Results

$$\langle W' \rangle$$

If the observable is zero then SUSY preserved



$W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{4}g\phi^4$

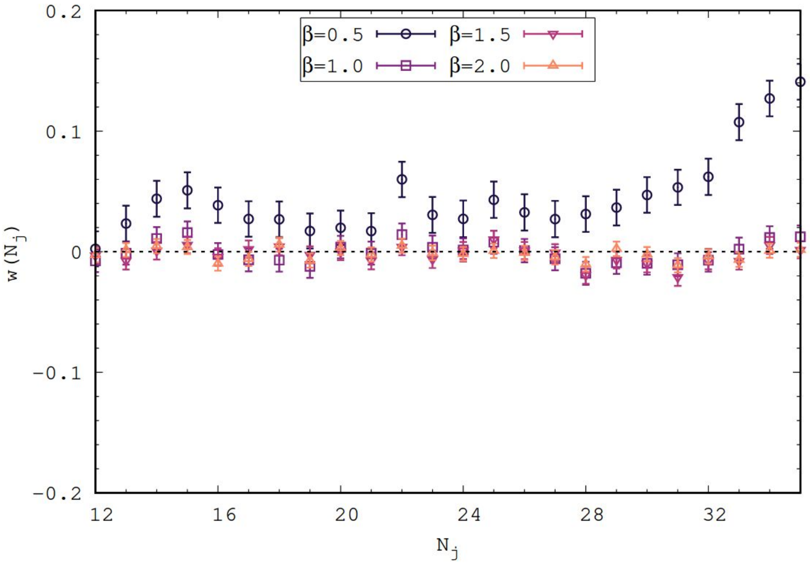


$W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{5}g\phi^5$

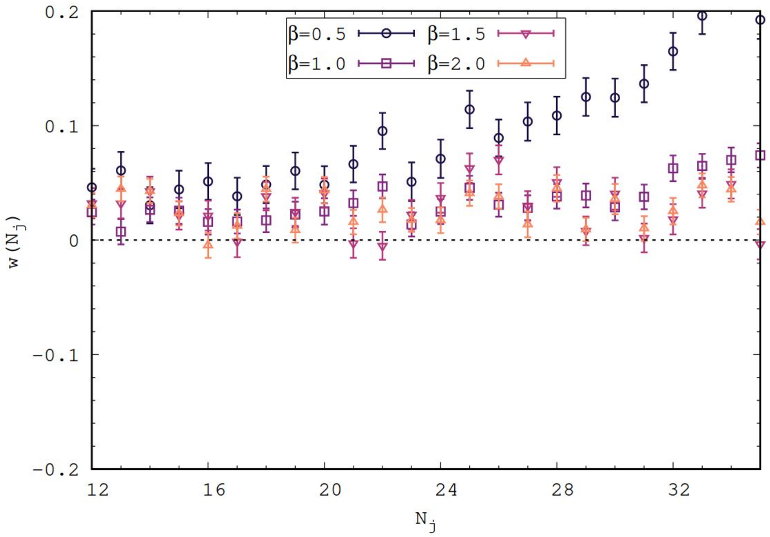
SUSYQM Results

$$w_1(n) \equiv \langle \phi_0 (D_{nk}\phi_k + W'_n) \rangle + \langle \overline{\psi}_n \psi_0 \rangle$$

If the observable fluctuates around zero then SUSY preserved



$W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{4}g\phi^4$



$W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{5}g\phi^5$

Takeaway SUSYQM

Potential	W'	SUSY Broken/Preserved
Degree 4	$W'(\phi) = m\phi + g\phi^3$	Preserved
Degree 5	$W'(\phi) = m\phi + g\phi^4$	Broken
Scarf	$W'(\phi) = \lambda\alpha \tan(\alpha\phi)$	Preserved
PT-symmetric*	$W'(\phi) = -ig\,(i\phi)^{1+\delta}$	Preserved

*For PT-symmetric superpotentials, only worked with even values of δ

- **Monte Carlo**
- **Lattice**
- **Supersymmetry**

Let us use combination of these for models with fields as matrices

SYM families

Lower dimensional SYM theories can be constructed by dimensionally reducing higher dimensional

$\mathcal{N} = 1$ SYM theories

16 supersymmetries
Maximal SYM family

$$\cancel{d = 10, \mathcal{N} = 1}$$
$$\cancel{d = 6, \mathcal{N} = 2}$$
$$\downarrow$$
$$d = 4, \mathcal{N} = 4$$
$$\downarrow$$
$$d = 3, \mathcal{N} = 8$$
$$\downarrow$$
$$d = 2, \mathcal{N} = (8, 8)$$
$$\downarrow$$
$$d = 1, \text{BFSS}$$
$$\downarrow$$
$$d = 0, \text{IKKT}$$

8 supersymmetries | 4 supersymmetries
Non-Maximal SYM families

$$\cancel{d = 6, \mathcal{N} = 1}$$
$$\cancel{d = 4, \mathcal{N} = 2}$$
$$\downarrow$$
$$d = 3, \mathcal{N} = 4$$
$$\downarrow$$
$$d = 2, \mathcal{N} = (4, 4)$$

$$\cancel{d = 4, \mathcal{N} = 1}$$
$$\cancel{d = 3, \mathcal{N} = 2}$$
$$\downarrow$$
$$d = 2, \mathcal{N} = (2, 2)$$

Lattice construction using 'twisting' requires 2^d supersymmetries github.com/daschaich/susy

- **MPI** based parallel code.

- Evolved from **MILC** code (which is developed by MIMD Lattice computation collaboration).

- Code is based on distributed memory systems. Can be tested on single-processor workstation or high performance computers.

- Performs **RHMC** simulations of SYM theories in various dimensions.

- Parallelization is between lattice sites, not on matrix degrees.

SUSY on Lattice

Lattice simulations of supersymmetric theories slightly complicated

- Broken SUSY on lattice
- Duality check requires runs at large *N*, computationally expensive
- Flat directions ➔ $[X_i, X_j] = 0$ ➔ but scalar eigenvalues keeps on increasing because of access to continuum branch of the spectra
- Sign problem ➔ Boltzmann factor e^{-S} cannot be used as weight in stochastic process

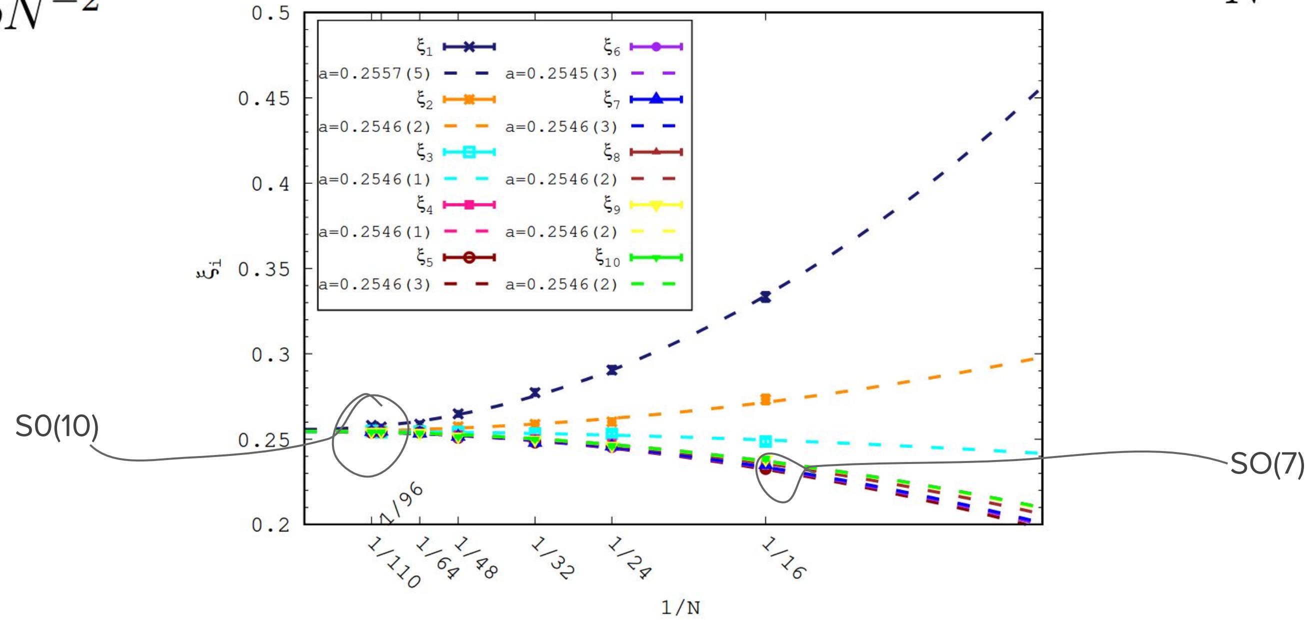
Finite N effects

$$S_{\mathrm{E}} = -\frac{N}{4\lambda}\sum_{i,j}\mathrm{Tr}([X^i, X^j]^2)$$

Will tune eigenvalues of a (10 x 10) matrix constructed out of scalars of bosonic IKKT model

$$I_{ij} = \frac{1}{N}\,\mathrm{Tr}(X^i X^j)$$

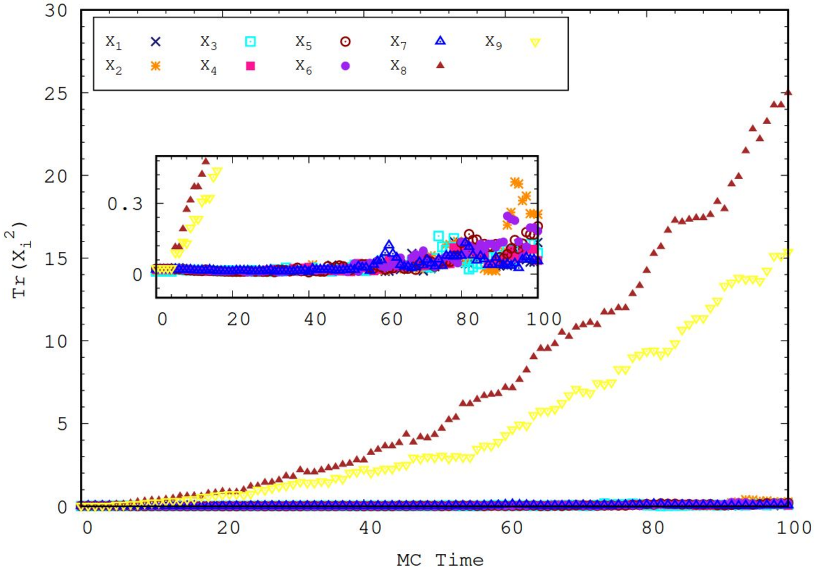
$$a + bN^{-2}$$



Flat directions

BFSS model

Runaway of scalars



This runaway can be controlled by:

- Adding a deformation term to the action and then fine-tuning it to recover target theory.
- By working with very large ***N***.


JHEP

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Non-perturbative phase structure of the bosonic BMN matrix model

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Abstract: We study the bosonic part of the BMN matrix model for wide ranges of temperatures, values of the deformation parameter, and numbers of colors $16 \le N \le 48$. Using lattice computations, we analyze phase transitions in the model, observing a single

JHEP05(2022)169

Let us start with one dimensional matrix model which is the BMN model, without fermions

Based on

JHEP **05**, (2022) 169 **NSD,** *Jha, Joseph, Samlodia, Schaich*

- No sign problem (as no fermions)
- No flat directions (model itself includes such deformation terms that controls these flat direction issues)
- Worked with different sizes of matrices to counter finite N effects

Matrix Models

BFSS Model

$$S_{\text{BFSS}} = \frac{N}{4\lambda}\int_0^\beta d\tau\ \mathrm{Tr}\Big\{ -(D_\tau X_i)^2 - \frac{1}{2}\sum_{i<j}[X_i, X_j]^2 + \Psi_\alpha^T \gamma^\tau_{\alpha\sigma} D_\tau \Psi_\sigma + \Psi_\alpha^T \gamma^i_{\alpha\sigma}[X_i, \Psi_\sigma]\Big\}$$

$$E/N^2 = 7.41 T^{14/5}$$

*PRD **58** (2016) 094501* *Hanada et al.*

Tested the gauge/gravity duality conjecture by computing the internal energy of the black hole directly from the gauge theory

Also provided stringy corrections to this Internal Energy

- SO(9) rotational symmetry

  A recent study using Gaussian expansion shows this symmetry broken like IKKT model
  arXiv:2209.01255 *Brahma, Brandenberger, Laliberte*

- Single deconfined phase in the theory

  A recent study with first results of confined phase
  *JHEP **05** (2022) 096* *Bergner et al.*

BMN Model

$$S_\mu = -\frac{N}{4\lambda}\int_0^\beta d\tau\ \mathrm{Tr}\left[\left(\frac{\mu}{3}X_I\right)^2 + \left(\frac{\mu}{6}X_A\right)^2 + \frac{\mu}{4}\Psi_\alpha^T\gamma_{\alpha\sigma}^{123}\Psi_\sigma - \frac{\sqrt{2}\mu}{3}\epsilon_{IJK}X_I X_J X_K\right]$$

- Mass deformed version of BFSS
- SO(9) explicitly broken into SO(6) X SO(3)
- First order phase transition

Different phases of the gravity dual

*JHEP **03** (2015) 069*

Costa, Greenspan, Penedones, Santos

Recent numerical studies to get these phases in gauge theories

*PoS **LATTICE21** (2022) 433*

Schaich, Jha, Joseph

Open: Other thermodynamic properties ??

BMN Model

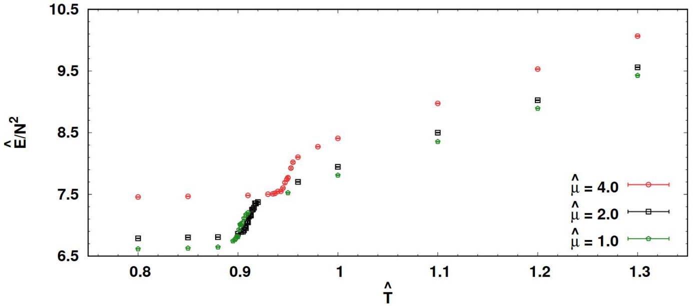
No fermions
➔ Clear deconfinement transition even in BFSS model

Easier to simulate
➔ Can work with large N setup

$$S_{\text{lat}} = \frac{N}{4\lambda_{\text{lat}}} \sum_{n=0}^{N_\tau - 1} \text{Tr}\left[-(\mathcal{D}_+ X_i)^2 - \frac{1}{2}\sum_{i<j}[X_i, X_j]^2 - \left(\frac{\mu_{\text{lat}}}{3} X_I\right)^2 - \left(\frac{\mu_{\text{lat}}}{6} X_A\right)^2 + \frac{\sqrt{2}\mu_{\text{lat}}}{3} \epsilon_{IJK} X_I X_J X_K \right]$$

Our setup

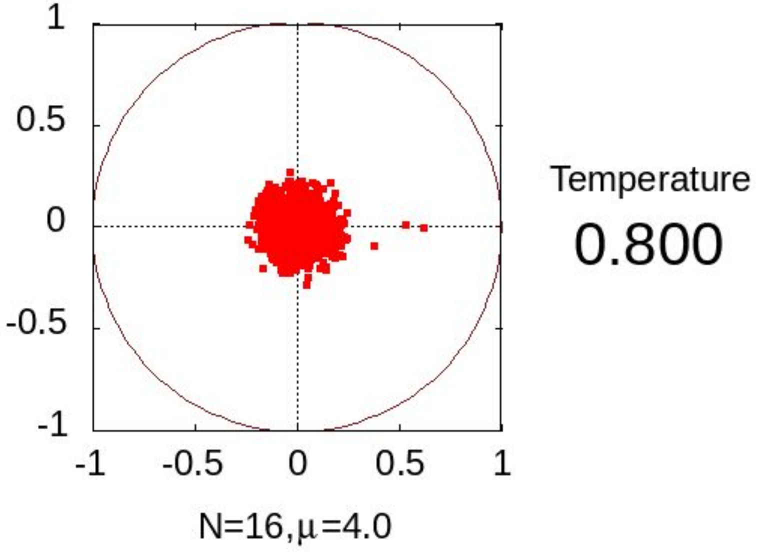
$$\frac{\widehat{E}}{N^2} \equiv \frac{E}{\lambda^{1/3} N^2} = \frac{1}{4N\lambda_{\text{lat}}^{4/3} N_\tau} \left\langle \sum_{n=0}^{N_\tau - 1} \text{Tr}\left(-\frac{3}{2}\sum_{i<j}[X_i, X_j]^2 - \frac{2\mu_{\text{lat}}^2}{9} X_I^2 - \frac{\mu_{\text{lat}}^2}{18} X_A^2 + \frac{5\sqrt{2}\mu_{\text{lat}}}{6} \epsilon_{IJK} X^I X^J X^K \right) \right\rangle$$

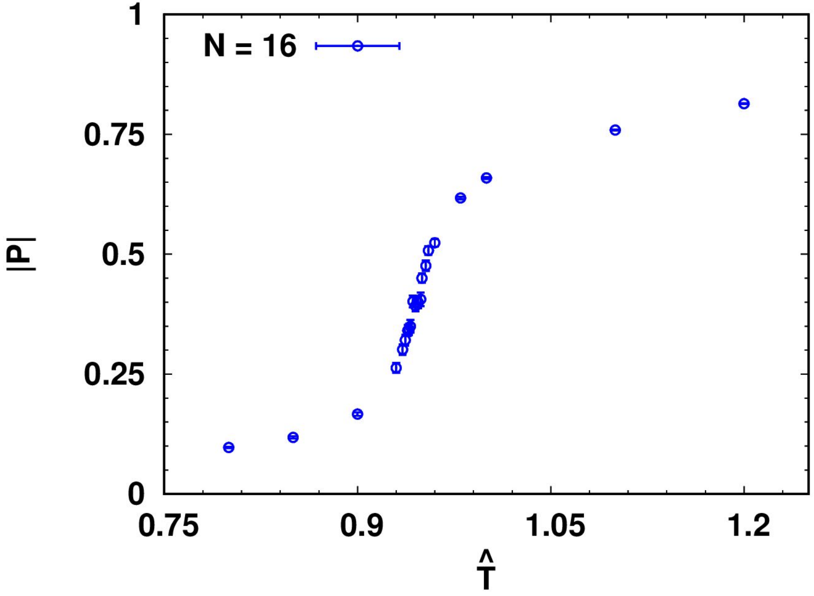


JHEP ***05*** *(2022) 169* **NSD**, Jha, Joseph, Samlodia, Schaich

Polyakov Loop

On lattice : $|P| = \left\langle \frac{1}{N} \left| \mathrm{Tr} \left(\prod_{n=0}^{N_\tau - 1} U(n) \right) \right| \right\rangle$





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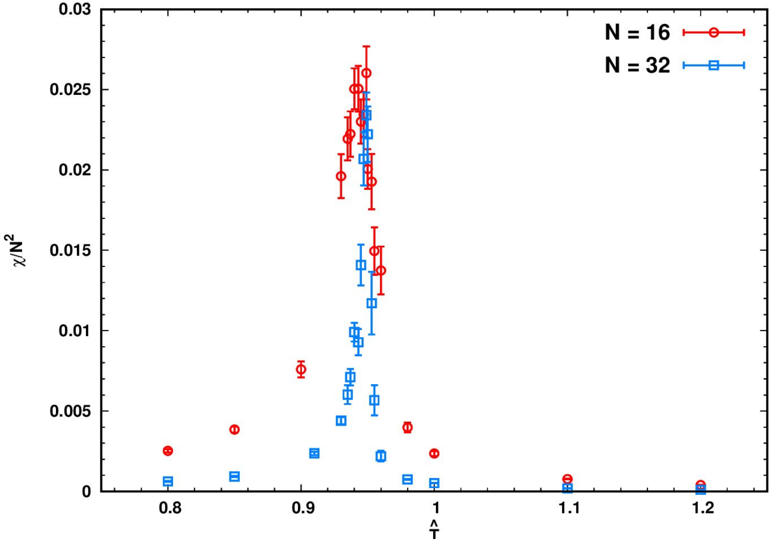
Transition Order

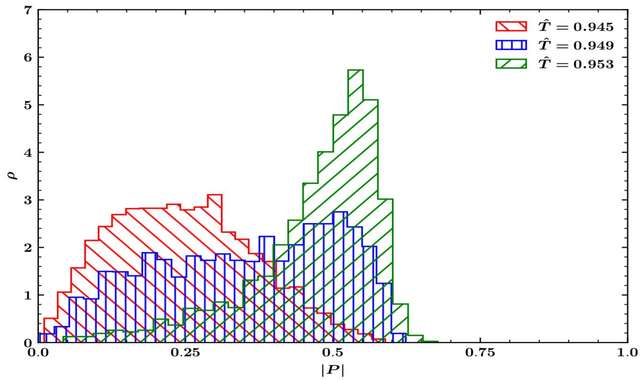
$$\chi \equiv N^2 \left(\langle |P|^2 \rangle - \langle |P| \rangle^2 \right)$$

- Susceptibility peaks at same height with ***N*²** normalization

- First order phase transition *PRL **113** (2014) 091603*

  Azuma, Morita, Takeuchi





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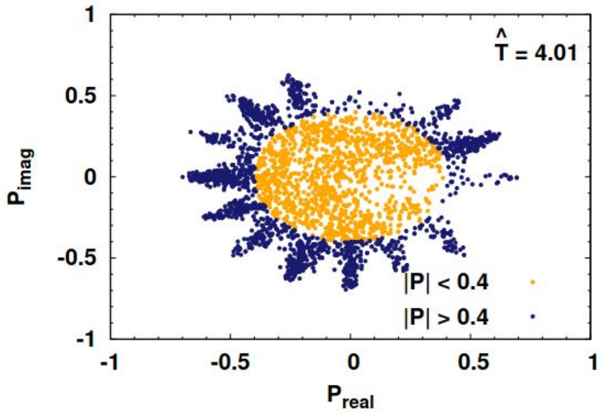
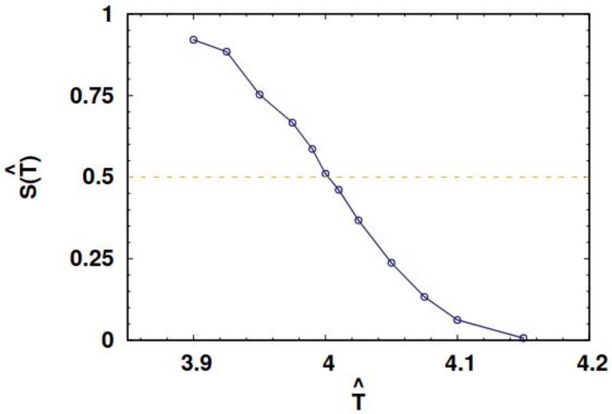
Separatrix Ratio

*PRD **91** (2015) 096002*

Francis, Kaczmarek, Laine, Neuhaus, Ohno





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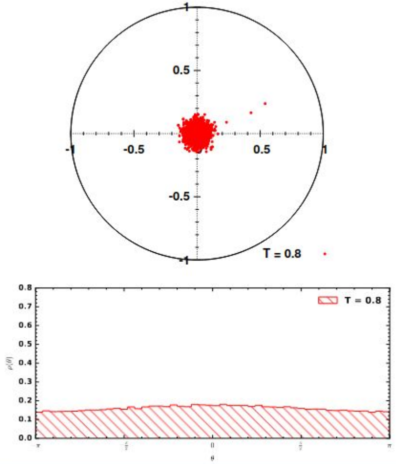
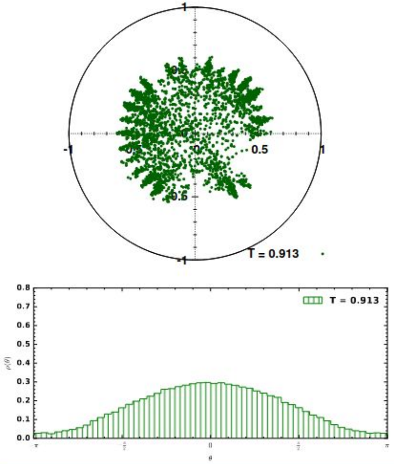
Different phases

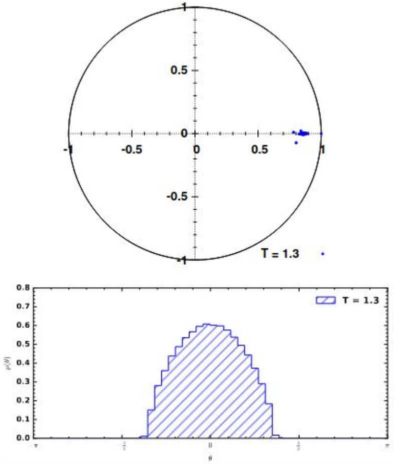
Angular distribution of Polyakov loop eigenvalues



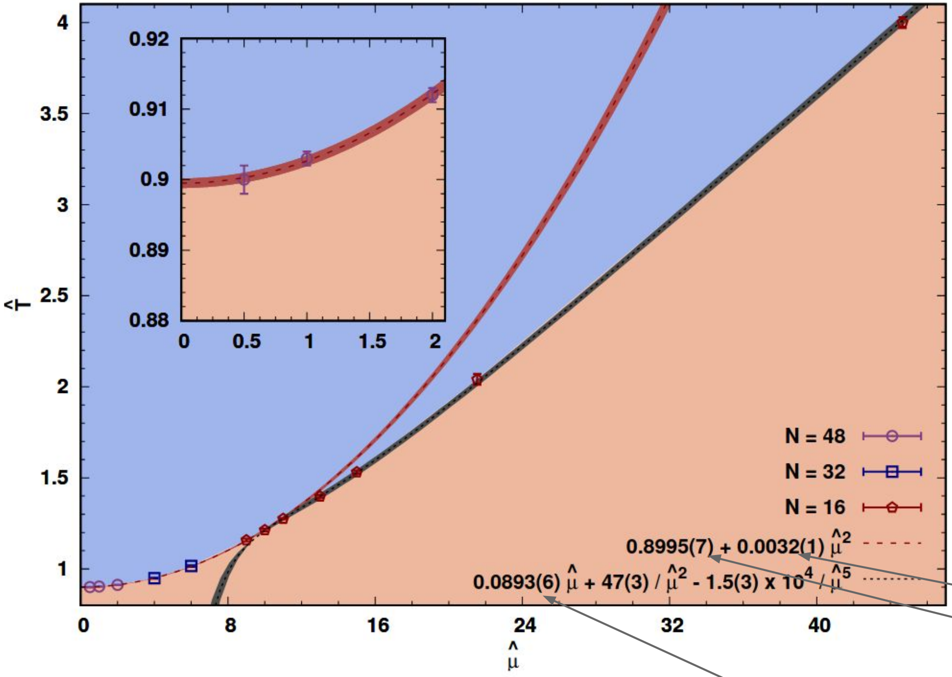
$T = 0.8, \mu_{\text{lat}} = 2.0$
Uniform phase



$T = 0.913, \mu_{\text{lat}} = 2.0$
Non-uniform phase



$T = 1.3, \mu_{\text{lat}} = 2.0$
Gapped phase



Phase Diagram

Perturbative calculation valid until $\mu \approx 10$, below it we enter strong coupling regime

First-order phase transition at all couplings

0.00330(2) *JHEP **05** (2022) 096*

0.8846(1) *Bergner et al.*

- Phase diagram smoothly interpolates between bosonic BFSS and gauged Gaussian limit

0.0893 *Adv.Theor.Math.Phys. **8** (2004) 603-696*
Aharony et al.

*JHEP **05** (2022) 169* **NSD**, Jha, Joseph, Samlodia, Schaich

Takeaway Bosonic BMN

- First order phase transition in the model at all values of couplings.
- Perturbative calculations valid upto a certain regime.
- Flat directions do not create any numerical problems, larger N required to get transition points for strong couplings.
- Numerical results smoothly interpolates between bosonic BFSS and gauged Gaussian limit.
- Separatrix method is a viable alternate option to investigate transition point.

Deconfinement transition in two-dimensional SU(N) Yang–Mills theory with four supercharges

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ABSTRACT: We study the large-N limit of two-dimensional Yang–Mills theory with four supercharges. Although this theory has no known holographic dual, we conduct numerical investigations to check for features similar to the sixteen-supercharge theory. We compute observables such as the gauge-invariant Wilson loops, energy density, the extent of scalars, and supersymmetric Ward identities with different lattice sizes and colors for a range of coupling. Our result suggests a possible deconfinement transition associated with the spatial Wilson loop, at large N, similar to the maximally supersymmetric case. However, the transition does not continue to strong coupling and potentially implies a lack of holographic interpretation for this minimally supersymmetric theory.

Let us start with move to slightly more complicated model. Two dimensional Yang-Mills with four supercharges including fermions.

Based on

arXiv:2303.xxxxx [hep-lat] (Under Preparation)

*PoS(**LATTICE2021**)433 (2022)*

*PoS(**LATTICE2022**)209 (2023) (In Press)*

***NSD,** Jha, Joseph, Schaich*

- No sign problem (in the region of interest)
- Numerical runaway due to flat directions (added explicit SUSY breaking terms to control runaway)
- Working with larger *N* more difficult as it is computationally costly, but got good results with sufficient *N* values

JHEP **07** (2013) 101 *Wiseman* ⬅️ About these peculiar powers from SYM

Lattice Results MSYM

For SYM theory in (1+p) dimensions

Bosonic action density $\propto t^{p+1}$, $t \gg 1$

$\propto t^{(14-2p)/(5-p)}$, $t \ll 1$

In conformal case both these cases are equivalent

Open: Deconfinement transition still needs numerical probing in this theory.

p = 2 PRD **102** (2020) 106009 *Catterall, Giedt, Jha, Schaich, Wiseman*

p = 1 PRD **97** (2018) 086020 *Catterall, Jha, Schaich, Wiseman*

2d $\mathcal{Q}$ = 4 SYM

Regularized on lattice using “**twisting**”

Another alternative is “**orbifolding**”

[Phys. Rept. 484 (2009) 71-130](#)

Catterall, Kaplan, Unsal

> Global symmetry:
>
> Four-dimensional theory
> $SO(4)_E \times U(1)$
>
> Two-dimensional theory
> $SO(2)_E \times SO(2)_{R_1} \times U(1)_{R_2}$

- Two possible twists possible as symmetry group contains two SO(2)’s

A $$\mathrm{SO}(2)' = \mathrm{diag}\Big(\mathrm{SO}(2)_E \times \mathrm{U}(1)_{R_2}\Big)$$

B ✓ $$\mathrm{SO}(2)' = \mathrm{diag}\Big(\mathrm{SO}(2)_E \times \mathrm{SO}(2)_{R_1}\Big)$$

2d $\mathcal{Q}$ = 4 SYM

Regularized on lattice using "**twisting**"

Another alternative is "**orbifolding**"

Phys. Rept. 484 (2009) 71-130

Catterall, Kaplan, Unsal

- Untwisted theory: 4 bosonic d.o.f., 4 fermionic d.o.f., 4 real supercharges
- Fermions, supercharges decomposed to integer spin representation and scalars, gauge fields combine to give complexified field
- Twisted theory: d.o.f. Fermions and complexified gauge field

$$\eta, \psi_a, \chi_{ab} \qquad\qquad \mathcal{A}_a$$

2d $\mathcal{Q} = 4$ SYM

$$\eta, \psi_a, \chi_{ab}$$

Fermions

- Obtained by dimensionally reducing $\mathcal{N} = 1$ SYM in 4d
- No holographic description

$$S = \frac{N}{4\lambda} \mathcal{Q} \int d^2x \, \mathrm{Tr} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \left[\overline{\mathcal{D}}_a, \mathcal{D}_a \right] - \frac{1}{2} \eta d \right)$$

$$[\mathcal{D}_a, \mathcal{D}_b]$$

$$\partial_a + \mathcal{A}_a$$

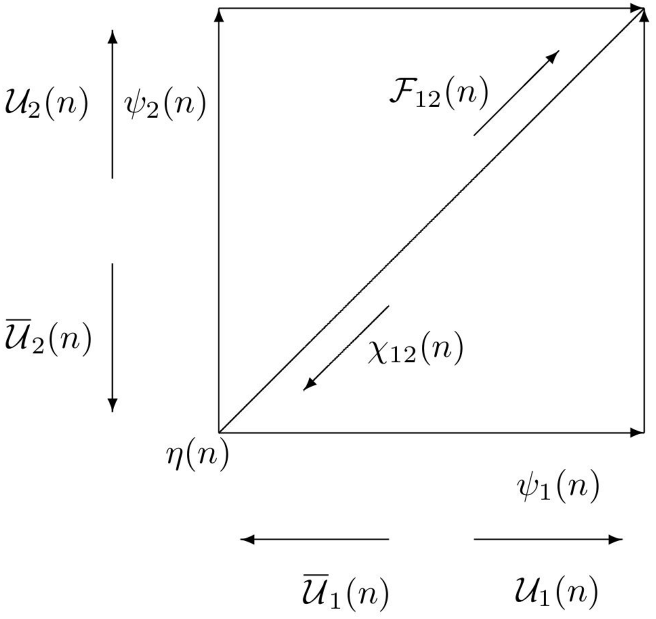
$$A_a + iX_a$$

$$\begin{aligned} \mathcal{Q}\mathcal{A}_a &= \psi_a, & \mathcal{Q}\overline{\mathcal{A}}_a &= 0, & \mathcal{Q}\psi_a &= 0, \\ \mathcal{Q}\chi_{ab} &= -\overline{\mathcal{F}}_{ab}, & \mathcal{Q}\eta &= d, & \mathcal{Q}d &= 0. \end{aligned}$$

After performing $\mathscr{Q}$ variation

2d $\mathcal{Q}$ = 4 SYM

$$S = \frac{N}{4\lambda}\int d^2x \ \mathrm{Tr}\left(-\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} + \frac{1}{2}\left[\overline{\mathcal{D}}_a, \mathcal{D}_a\right]^2 - \chi_{ab}\mathcal{D}_{[a}\ \psi\ _{b]} - \eta\overline{\mathcal{D}}_a\psi_a \right)$$



On Lattice

- Gauge field → Wilson link $\mathscr{A}_a(x) \rightarrow \mathscr{U}_a(n)$, on links of square lattice
- To preserve SUSY $\psi_a(n)$ lives on same links as bosonic superpartners
- $\eta(n)$ associated with site
- $\chi_{ab}(n)$ lives on diagonal

$$S = \frac{N}{4\lambda_{\mathrm{lat}}}\sum_n \mathrm{Tr}\left[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) + \frac{1}{2}\left(\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n)\right)^2 - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi\ _{b]}(n) - \eta(n)\overline{\mathcal{D}}_a^{(-)}\psi_a(n)\right],$$

Simulation setup

- To control flat directions

$$S_{\text{total}} = S + \frac{N\mu^2}{4\lambda_{\text{lat}}} \sum_{n,a} \text{Tr}\left(\overline{\mathcal{U}}_a(n)\mathcal{U}_a(n) - \mathbb{I}_N\right)^2$$

- Worked with different mass deformations

$$\mu = \zeta \frac{r_\tau}{N_\tau} = \zeta\sqrt{\lambda}\mathrm{a} = \zeta\sqrt{\lambda_{\text{lat}}}$$

- Different aspect ratio lattices

$$\alpha \equiv \frac{r_x}{r_\tau} = \frac{N_x}{N_\tau}$$

- Different gauge groups, anti-periodic boundary conditions for fermions

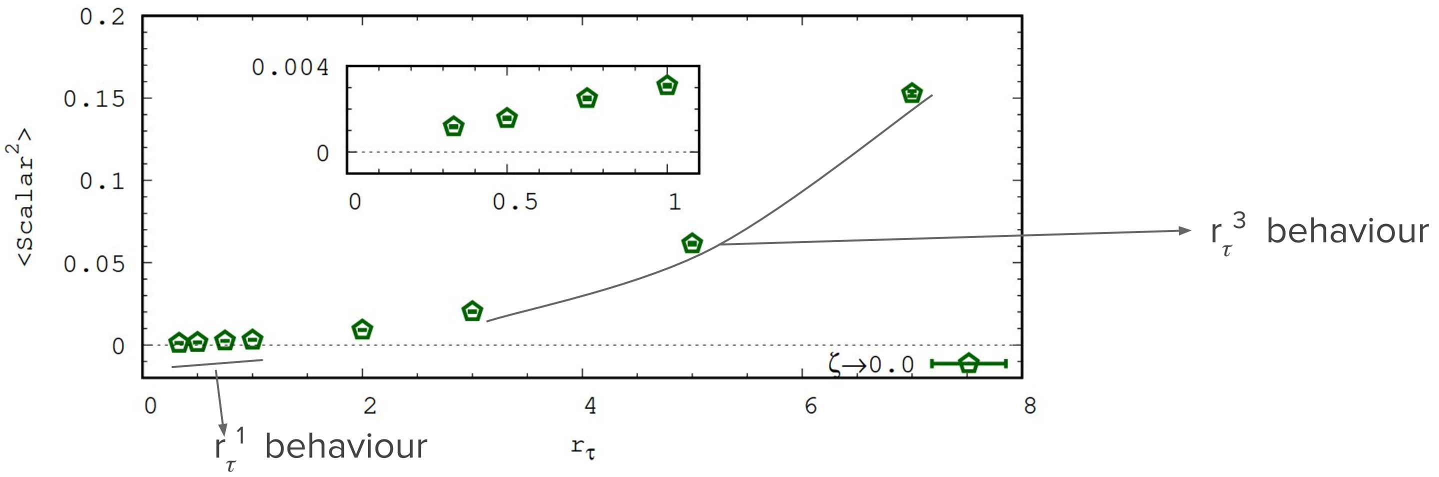
Lattice Results

Scalar² → Tr (X^2)
24 x 24 lattice, N =12

*JHEP **07** (2013) 101*

Wiseman

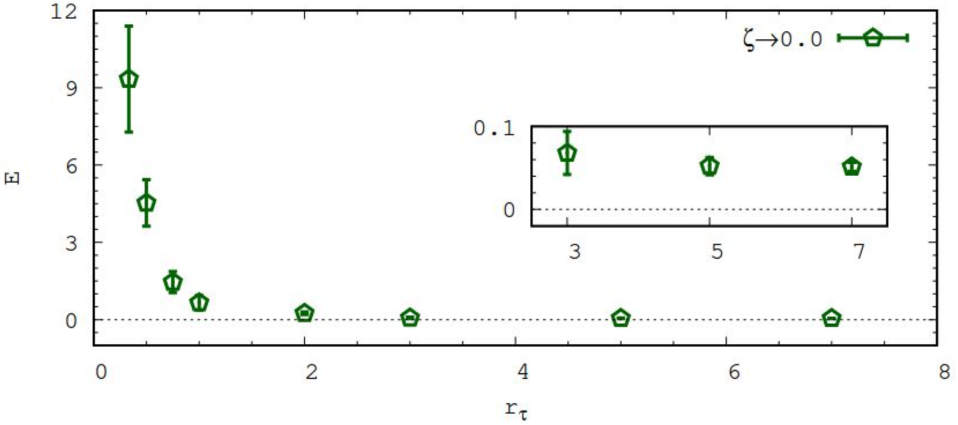
- Behaviour different than maximal cousin
- Existence of bound state at finite temperature



(To appear soon) **NSD**, Jha, Joseph, Schaich

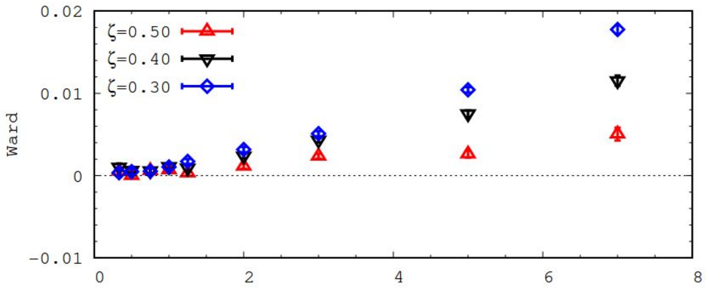
Lattice Results

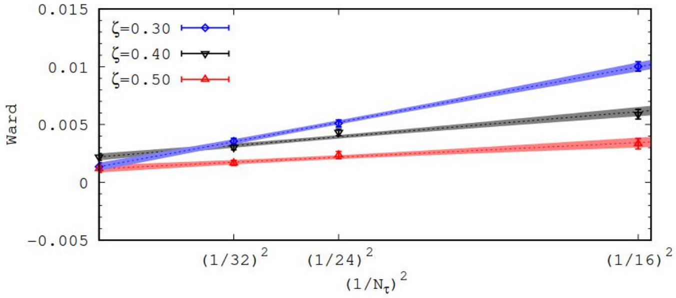
Preserved SUSY
24 x 24 lattice, N =12



$$E = \frac{3}{\lambda_{\mathrm{lat}}}\left(1 - \frac{2}{3N^2}S_B\right)$$

$$\mathcal{Q}\sum_a \left(\eta\,\mathcal{U}_a\bar{\mathcal{U}}_a\right)$$





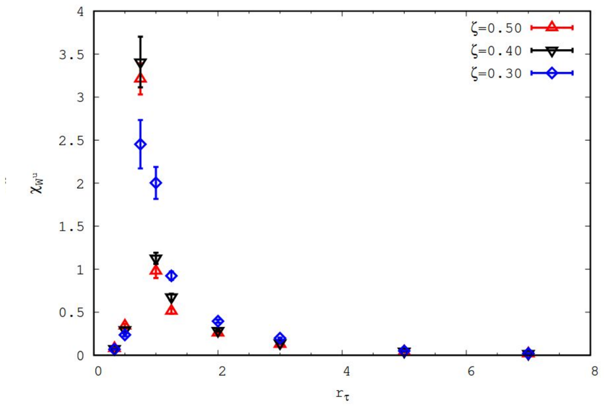
(To appear soon) **NSD**, Jha, Joseph, Schaich

Lattice Results

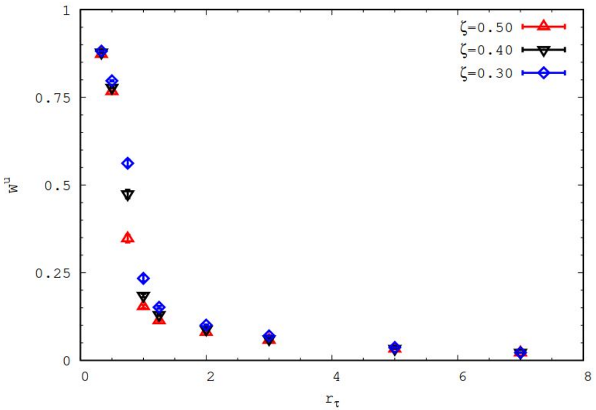
Spatial deconfinement transition

24 x 24 lattice, N =12

Wilson loop along temporal and spatial direction

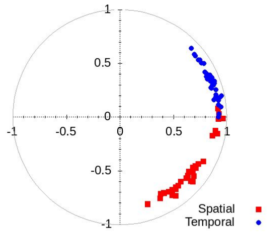


Variance of spatial WL

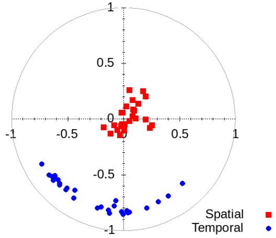


MC time history

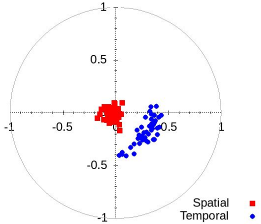
r_τ=0.5,ζ=0.3



r_τ=1.0,ζ=0.3



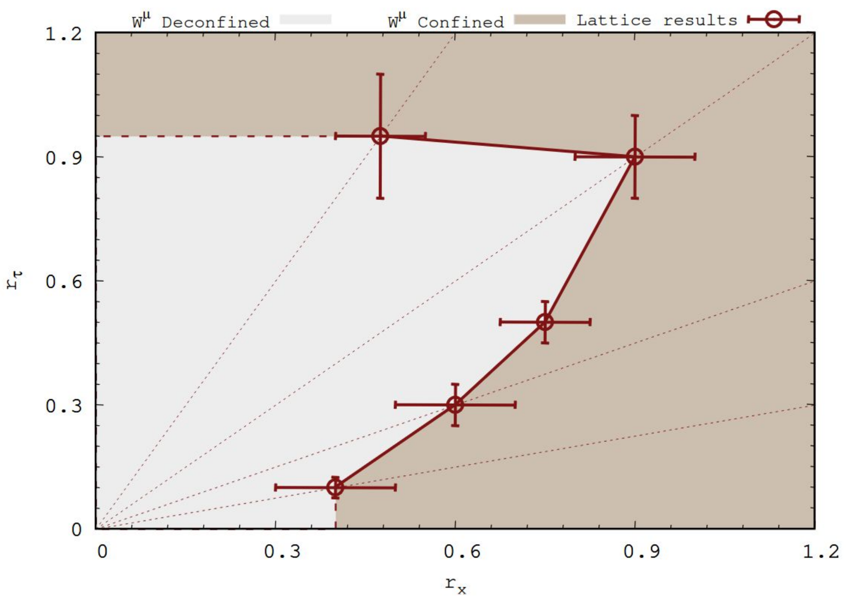
r_τ=3.0,ζ=0.3

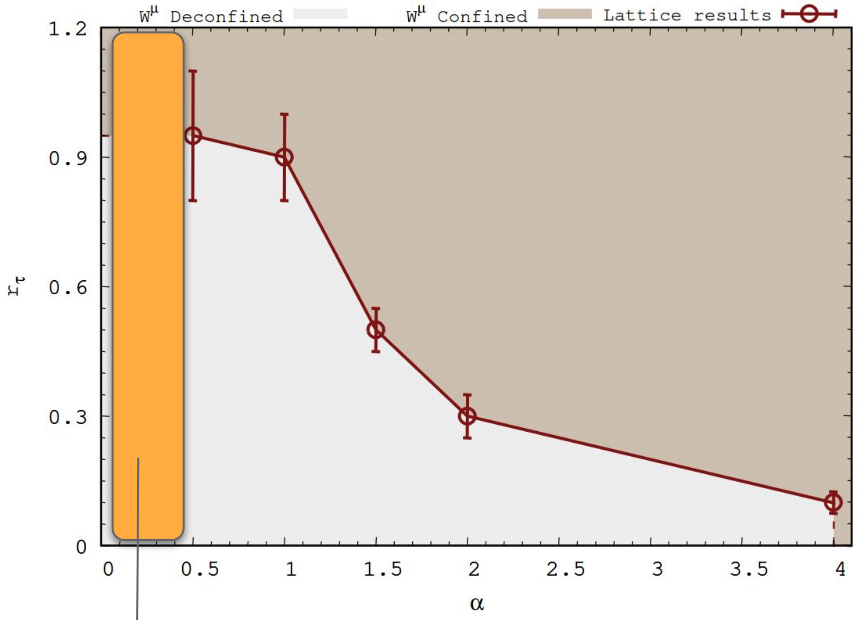


(To appear soon) **NSD**, Jha, Joseph, Schaich

Lattice Results

Phase diagram
Different aspect ratio **α**, N =12





Problematic regime in numerical simulations

(To appear soon) **NSD**, Jha, Joseph, Schaich

Takeaway 2d $\mathcal{Q} = 4$ SYM

- Scalars show bound state behaviour
- Spatial deconfinement transition, but only limited to weak coupling regime
- Thermodynamics different than maximal counterpart
- More analysis required to probe if it admits holographic description : **Open**

Future Directions

- Numerical tools beyond Monte Carlo, especially for lower dimensional models
  - Numerical bootstrap is a viable option to investigate Matrix Models *JHEP **06** (2020) 090* *Lin*
- Numerically investigating non-gauge/gravity *JHEP **04** (2018) 084* *Maldacena, Milekhin*
  - Recent numerical results *JHEP **08** (2022) 178* *Pateloudis et al.*
- Continue exploring non-maximal and maximal supersymmetric theories
- Improving Monte Carlo Method

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- Arpith, Vamika, Minati, Bana
- MS people
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- Aakash, Aditya, Ankit, Harkirat, Manpreet, Pankaj, Rajeev, Sachin, Sheenam, Subhashree, Warseem

To MOM

THANK YOU

Numerical Bootstrap

- To derive the spectrum of the theory by checking the positivity of some of the observables.

    - Taking the help of loop equations to connect various orders of observables.

$$\mathcal{M} = \begin{bmatrix} \langle O_0^\dagger O_0 \rangle & \langle O_0^\dagger O_1 \rangle & \cdots & \langle O_0^\dagger O_K \rangle \\ \langle O_1^\dagger O_0 \rangle & \langle O_1^\dagger O_1 \rangle & \cdots & \langle O_1^\dagger O_K \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle O_K^\dagger O_0 \rangle & \langle O_K^\dagger O_1 \rangle & \cdots & \langle O_K^\dagger O_K \rangle \end{bmatrix} \geq 0$$

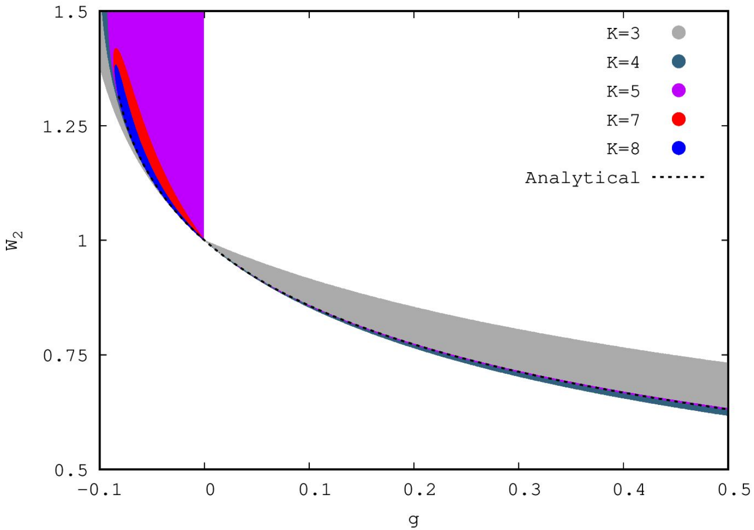
Numerical Bootstrap

$$V = m\frac{X^2}{2} + g\frac{X^4}{4}$$

$$mW^n + gW^{n+2} = \sum_{j=0}^{n-2} W^j W^{n-2-j}$$

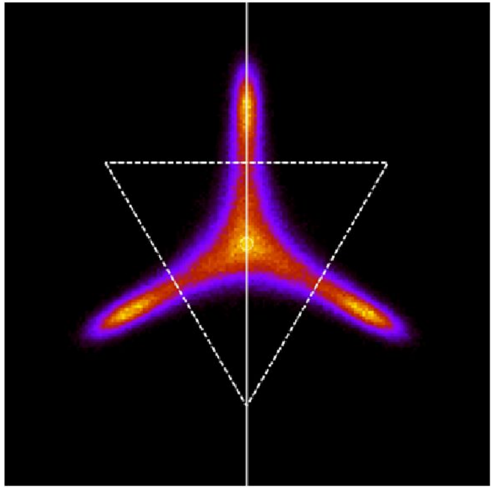
$$\left\langle \frac{1}{N}\mathrm{Tr}\left(X^2\right)\right\rangle = \frac{(12g+m^2)^{1.5} - 18mg - m^3}{54g^2}$$

$$\mathcal{M} = \begin{bmatrix} \langle X^0\rangle & \langle X^1\rangle & \langle X^2\rangle & \cdots & \langle X^K\rangle \\ \langle X^1\rangle & \langle X^2\rangle & \langle X^3\rangle & \cdots & \langle X^{K+1}\rangle \\ \vdots & \vdots & \ddots & \vdots & \\ \langle X^K\rangle & \langle X^{K+1}\rangle & \langle X^{K+2}\rangle & \cdots & \langle X^{2K}\rangle \end{bmatrix} \geq 0$$

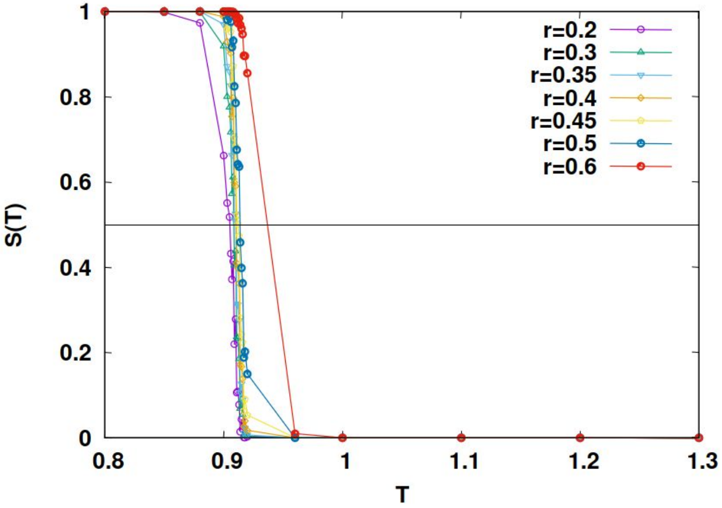


- This plot generated in less than 1 minute.
- But gets complicated as number of matrices increase

Separatrix

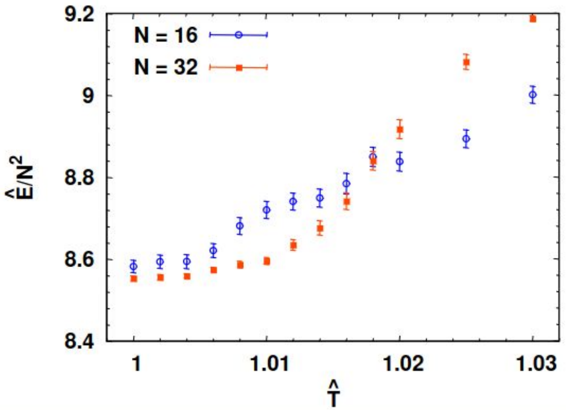


PRD 91 (2015) 096002

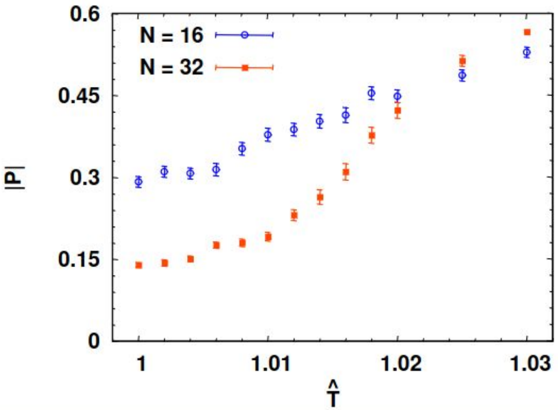


Separatrix ratio vs r $N = 32, \hat{\mu} = 2$

BBMN Results



Energy $\hat{\mu} = 6$



Polyakov Loop $\hat{\mu} = 6$

First order transition

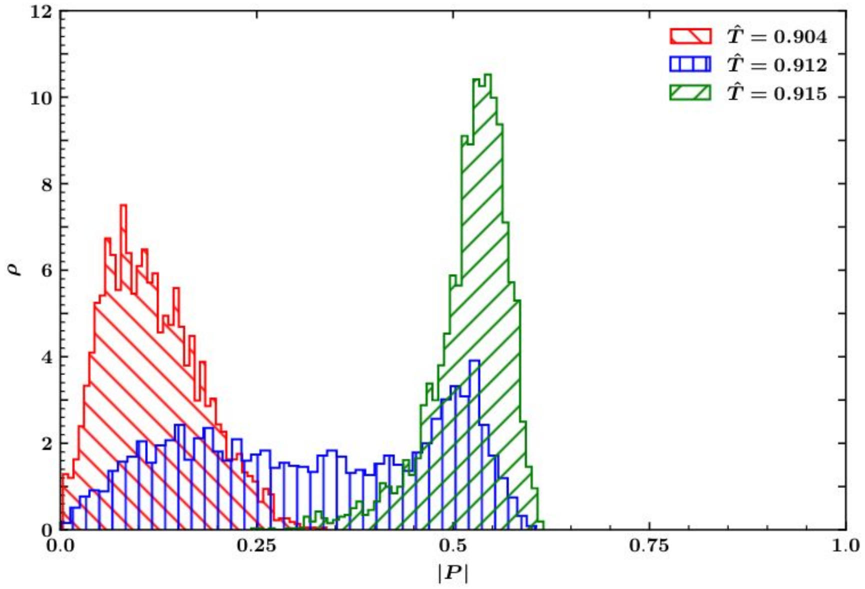


FIGURE 4.12: Polyakov loop magnitude distribution at three different temperatures for $\widehat{\mu} = 2.0$ with $N = 48$. A two-peak structure appears to develop more clearly as compared with lower N values.

AP BC Fermions

Thermal green function

$$G_B(x,y,\tau_1,\tau_2) = Z^{-1} Tr\left[e^{-\beta K} T\left[\hat{\phi}(x,\tau_1)\hat{\phi}(y,\tau_2)\right]\right]$$

using step fn. with $\tau_1 = \tau$, $\tau_2 = 0$ and cyclic property of trace

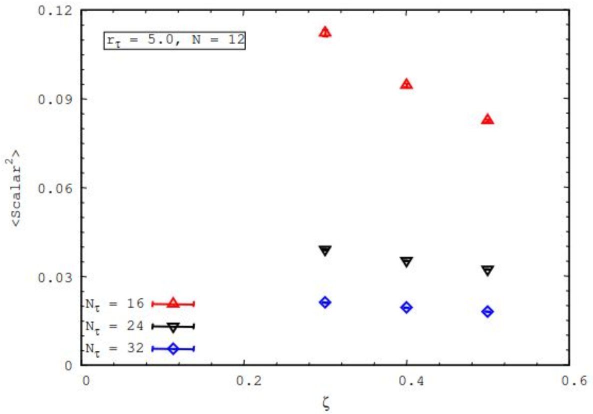
$$G_B(x,y,\tau,0) = Z^{-1} Tr\left[\hat{\phi}(y,0)e^{-\beta K}\hat{\phi}(x,\tau)\right]$$

$$G_B(x,y,\tau,0) = Z^{-1} Tr\left[e^{-\beta K}e^{+\beta K}\hat{\phi}(y,0)e^{-\beta K}\hat{\phi}(x,\tau)\right]$$

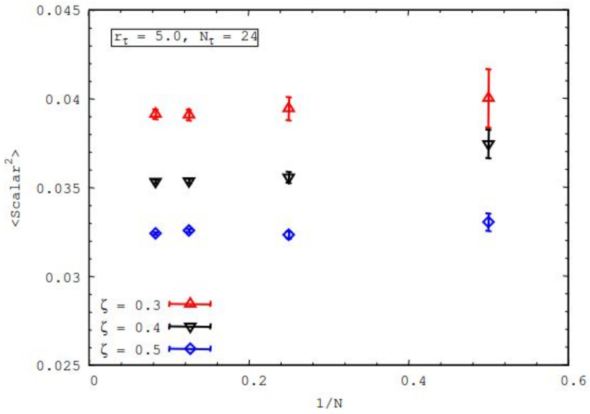
$$G_B(x,y,\tau,0) = Z^{-1} Tr\left[e^{-\beta K}\hat{\phi}(y,\beta)\hat{\phi}(x,\tau)\right]$$

If ϕ's are bosons last two interchanged gives $\phi(y,\beta) = \phi(y,0)$, if ϕ's are fermions (say ψ) last two interchanged gives extra -ve sign $\psi(y,\beta) = -\psi(y,0)$, hence APBC for fermions

Bound state 2d

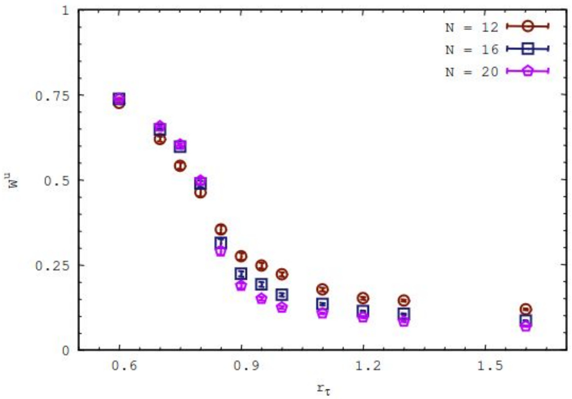


Bound state vs lattice size

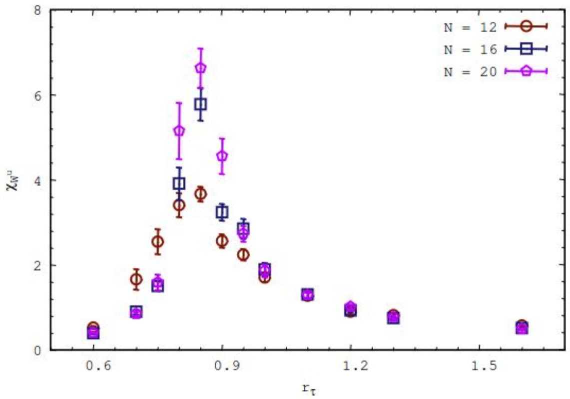


Bound state vs gauge group

Transition order 2d

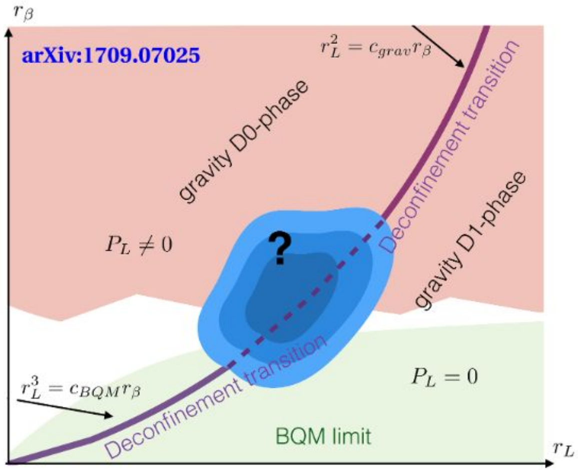


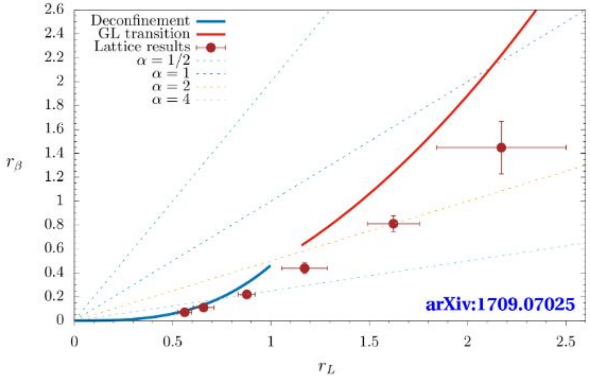
Wilson loop dependence on N



χ vs N hints second order phase transition

Maximal theory 2d





$\mathcal{Q} = 16$

Ward Identity

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi' \mathcal{O}(\phi') e^{-S(\phi')}$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O}(\phi) e^{-S(\phi)} [1 - \delta S(\phi)] + \frac{1}{Z} \int \mathcal{D}\phi \delta \mathcal{O}(\phi) e^{-S(\phi)} [1 - \delta S(\phi)].$$

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle - \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O}(\phi) \delta S(\phi) e^{-S(\phi)} + \frac{1}{Z} \int \mathcal{D}\phi \delta \mathcal{O}(\phi) e^{-S(\phi)}.$$

As action is invariant under infinitesimal transformation δ
Hence $\langle \delta \mathcal{O} \rangle = 0$

Energy spectrum

Response: The action in (2.1) is

$$S = \int_0^\beta d\tau \left(-\frac{1}{2}\phi\partial_\tau^2\phi + \overline{\psi}\partial_\tau\psi - \frac{1}{2}B^2 + \overline{\psi}W''(\phi)\psi - BW'(\phi) \right). \tag{28}$$

After integrating out the auxiliary field, we can write the Hamiltonian operator of the action as

$$H = \frac{1}{2}\begin{pmatrix} H_B & 0 \\ 0 & H_F \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\partial_x^2 + W'^2 - W'' & 0 \\ 0 & -\partial_x^2 + W'^2 + W'' \end{pmatrix}, \tag{29}$$

Now depending upon the form of W, we can tell whether the ground state forms a singlet or if it is degenerate. For the simplest superpotential derivative $W' = m\phi$, we can clearly see that in the bosonic sector, the energy states are $0, m, 2m, \ldots$, and in the fermionic sector, the energy states are $m, 2m, 3m, \ldots$.

SUSY Tn.

Response: Let us try to show an example of how these supersymmetries keep the action invariant. For simplification purposes, we will start from the continuum action as

$$\delta_2 S = \int d\tau\ \delta_2 \left(-\frac{1}{2}\phi\partial_\tau^2\phi + \overline{\psi}\partial_\tau\psi + \overline{\psi}W''(\phi)\psi + \frac{1}{2}\left[W'(\phi)\right]^2\right). \tag{30}$$

The dynamical ϕ term in the action can be integrated, and it takes the form $\frac{1}{2}(\partial_\tau\phi)^2$. Now, operating the supersymmetry transformations on the resultant equation

$$\begin{aligned}
\delta_2 S &= \int d\tau\ \delta_2 \left(\frac{1}{2}(\partial_\tau\phi)^2 + \overline{\psi}\partial_\tau\psi + \overline{\psi}W''(\phi)\psi + \frac{1}{2}\left[W'(\phi)\right]^2\right), \\
&= \int d\tau\ \left((\partial_\tau\phi)\delta_2(\partial_\tau\phi) + \delta_2(\overline{\psi}\partial_\tau\psi + \overline{\psi}W''(\phi)\psi) + W'(\phi)\delta_2(W'(\phi))\right), \\
&= \int d\tau\ \left(\partial_\tau\phi\,\bar{\epsilon}\,\partial_\tau\psi - \bar{\epsilon}(\partial_\tau\phi + W')(\partial_\tau\psi + W''(\phi)\psi) + W'(\phi)W''(\phi)\bar{\epsilon}\psi\right).
\end{aligned} \tag{31}$$

One term in the above equation is not listed, which is operating the transformation on the second derivative of the superpotential, as it gives ψ after operating the supersymmetry, and $\psi^2 = 0$. Now let us expand the above equation

$$\begin{aligned}
\delta_2 S &= \int d\tau\ \bar{\epsilon}\left(\partial_\tau\phi\,\partial_\tau\psi - (\partial_\tau\phi + W')(\partial_\tau\psi + W''(\phi)\psi) + W'(\phi)W''(\phi)\psi\right), \\
&= \int d\tau\ -\bar{\epsilon}\left((\partial_\tau\phi W''(\phi)\psi + W'\partial_\tau\psi\right), \\
&= \int d\tau\ \partial_\tau\left(-\bar{\epsilon}W'(\phi)\psi\right).
\end{aligned} \tag{32}$$

Fermion doubling

Dirac propagator free theory:

$$S = \frac{m - ia^{-1}\sum_\mu \gamma^\mu \sin(p^\mu a)}{m^2 + a^{-2}\sum_\mu \sin(p^\mu a)^2}$$

For low momenta pole at $p^\mu a = (am, 0, 0, 0)$

But fifteen additional poles at $p^\mu a = (am, 0, 0, 0) + \pi^\mu$

As $sin(p^\mu a)$ has two poles in range $p^\mu = [-\pi/a, \pi/a]$

SUSY 2d Minkowski Action

$$S = \frac{1}{2g^2}\int d^2x\ \mathrm{Tr}\Big\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + i\overline{\lambda}\Gamma^\mu D_\mu\lambda + D_\mu\phi_m D^\mu\phi_m$$
$$+\overline{\lambda}\Gamma^{m+1}[\phi_m,\lambda] + \frac{1}{2}[\phi_m,\phi_n][\phi^m,\phi^n]\Big\}.$$

β function - Backup

$$\beta(g) = \frac{\partial g}{\partial \log(\mu)}$$

- g : coupling parameter, μ : energy scale
- β vanishes at particluar g, scale invariant
- $\mathcal{N} = 4$ SYM, all beta functions vanish, energy-momentum tensor traceless, charge associated with CT preserved, Conformal
- Scale invariance not all β functions vanish

$$\beta_{\text{QED}} = \frac{e^3}{12\pi^2}$$

$$\beta_{\text{QCD}} = -\left(-11 - \frac{n_s}{6} - \frac{2n_f}{3}\right)\frac{g^3}{16\pi^2}$$

- $n_f < 16$ coupling increases with decrease in energy scale, no longer rely on perturbation

$$T_{\mu\nu} = \frac{\partial S}{\partial g^{\mu\nu}}$$

For YM

$$T^{\mu}_{\mu} = \left(\frac{g^{\mu}_{\mu}}{2} - 1\right)|F|^2$$

Only zero in 4d as trace of metric is 2 in it